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ALGEBRA

FOR

MIDDLE OR LOWER SECONDARY SCHOOLS

WITH NEARLY

ONE THOUSAND EXAMPLES

CAREFULLY GRADED FOR A SYSTEMATIC COURSE

BY

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PREFACE TO THE FIFTH EDITION.

THIS book originally appeared as a Class-Book of Algebra Examples. Its distinctive feature was the division of each Exercise into three sets A, B, C of equal difficulty, to be given to contiguous pupils in a class, and to thereby insure independent work. While this has been retained as a valuable auxiliary to teaching, it seems to the author that the interweaving of the explanations of the algebraic processes with the illustrative Exercises will emphasise the value of the former, and discourage mere mechanical working. For beginners in algebra neatness of work is of prime importance. Hence the number of examples in each Exercise has been reduced, in order that more attention may be given to neatness of style and to arrangement.

Particular attention has been paid to Factorisation, and to Multiplication and Division at Sight: as the after progress of the student is to a great extent dependent on his power to employ these processes readily.

In this Fifth Edition a few misprints found in the Fourth Edition have been corrected; but otherwise the work is the same.

J. COOK.

BANGALORE,
24th March 1897.

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GENERAL RULES FOR THE STUDENT OF ALGEBRA.

i. The working of an example should always show **clearly** how each step follows from the preceding one; and for this the correct use of short explanatory words or symbols is of much importance.

ii. **Neatness of arrangement** in Algebra is of the greatest value, and in itself goes far to insure accuracy.

iii. The sign = (equal) should only be used in its correct sense, to connect quantities or expressions which are equal; and the several steps of a solution should be arranged so that the signs of equality are placed one under the other.

iv. It is better not to *cancel* in Algebra *by drawing lines* as is done in Arithmetic; if the steps of the solution are clear, the elimination of a factor, term, &c., will be obvious without this.

Form II 03)

ALGEBRA FOR MIDDLE SCHOOLS.

Chap. I.—ALGEBRAIC SIGNS AND THEIR USE.

§ 1. **A Quantity** is a measure of any kind. In Arithmetic all *quantities* are special or particular, that is, have definite numerical values: as 3 pounds, 4 yards, &c., which are **concrete** arithmetical quantities, *i.e.*, definite measures of definite things; or 3, 4, &c., which are said to be **abstract** arithmetical quantities, which only tell the number of times the unit—whatever it may be—is repeated, but tells nothing of the nature of the unit referred to.

§ 2. In **algebraic quantities** no particular numerical values belong to the symbols used; they are general quantities and may have one value in one question and another value in another. They are usually denoted by the **letters of the alphabet** a, b, c, &c., or x, y, z. They might however be denoted by the colours of the rainbow, or by any other convenient symbols: thus the fourth proposition of Euclid, Book III, might be symbolically rendered as follows: Let a line be divided into two parts, one called red, and the other blue; then square of the whole line (red + blue) = square of the red + square of the blue + twice the product of red into blue; or briefly $(r + b)^2 = r^2 + b^2 + 2rb$. This shows the advantage of algebraic brevity by the use of symbols. The eye and the mind can take in at a glance a statement which is obscured by the many words that are otherwise needed to express it.

§ 3. **The algebraic signs** most commonly used are the following :—

+ , which is read **plus** (Lat. = *more*), is the sign of **addition** ; 3 + 4, or 3 *plus* 4, means 3 more than 4, *i.e.*, 7.

—, which is read **minus** (Lat. = *less*), is the sign of **subtraction** ; 4 — 2, or 4 *minus* 2, means 4 less 2, *i.e.*, 2.

± , which is read **plus or minus,** means the **sum or difference of** ; 5 ± 3, or 5 *plus or minus* 3, is 8 or 2.

× , or . , which is read **into,** or **multiplied by,** is the sign of **multiplication** ; 3 × 4, or 3. 4, is read 3 *into* 4, or 3 *multiplied by* 4, *i.e.*, 12. This multiplication dot is written *on* the line ; the decimal dot *above* the line.

÷ , or /, which is read **divided by,** is the sign of **division** ; 8 ÷ 2, or 8/2, means 8 *divided by* 2, *i.e.*, 4.

( ), { }, [ ], are called **brackets** [diminutive of *brace*, from Lat. *brachium* = an arm] and are small signs to hold all enclosed within them together as a single number or quantity.

( ) are called **common brackets** because most frequently used.

{ } are called **face brackets,** from their appearance.

[ ] are called **square brackets,** and are generally used to include a large number of separate quantities.

——, drawn over two or more letters or numbers, is called a **vinculum** [Lat. = a *chain* or *bond*], and has the same force as a pair of common brackets.

$\sqrt{\ }$ is called **the radical** or **root sign,** and is merely a large form of the letter **r.**

$\sqrt{16}$ denotes the square root of 16, *i.e.*, 4; $\sqrt[3]{27}$ denotes the third root, or cube root as it is called, of 27, *i.e.*, 3; $\sqrt[4]{81}$ denotes the fourth root of 81, *i.e.*, that number which multiplied by itself 4 times gives 81, *i.e.*, 3.

$=$, which is read **is equal to,** denotes that the two numbers or quantities between which it is placed, are of equal value, *e.g.*, $4 + 3 = 1 + 6$.

$\therefore$ is used as a short-hand symbol for **therefore,** and is used at the beginning of a new step in reasoning, and generally at the beginning of a line.

$\because$ is used in the same way as short-hand for **because,** and is used only at the beginning of a line in a train of reasoning.

$>$ is used to mean **is greater than;** and $<$ to mean **is less than;** *e.g.*, $4 > 3$, and $3 < 4$; in each case the opening is towards the greater number.

[*N.B.*—It is an error to use the signs $=$, $>$, $<$, for the adjectives *equal*, *greater*, *less*; they have a predicative force.]

§ 4. Elementary Definitions.

An algebraic expression is the symbol or collection of symbols used to denote any quantity or quantities.

The Terms are the parts of an expression separated by the signs $+$ or $-$.

An expression is called **monomial** if it has only *one term*; **binomial** if it has only *two terms*; **trinomial** if it has only *three terms*; and **polynomial** or **multinomial** if it has *more than three terms.*

A simple expression has only *one term*; a **compound** or **complex expression** has several terms.

The Product is the result of multiplying two or more quantities together and is denoted in Algebra thus:—$a \times b$, or $a.\,b$, or simply ab.

A Factor is one of the quantities which go to make up a product; thus a and b are the two factors of the product ab.

The Quotient is the result of dividing one quantity by another, and is denoted thus:—$a \div b$, or $\frac{a}{b}$ or a/b.

The Co-efficient is a factor of a quantity; *e.g.*, in $2x$, 2 is the co-efficient. Sometimes a letter may be taken as a co-efficient, *e.g.*, ax; in this case a is called a **literal co-efficient** [Lat. *litera* = a letter]; while 2, in the former case, is, by way of distinction, called a **numerical co-efficient.** Most frequently the co-efficient is partly numerical and partly literal, *e.g.*, $3ax$; here the co-efficient is $3a$.

A Power of a quantity is the number of times that any quantity is multiplied by itself to form a product; *e.g.*, $r \times r$ is called the second power or square of r; so $r \times r \times r$ is the cube or third power of r; and the first great step in Algebra was made when in 1637 Descartes denoted the product of $r \times r \times r$ by r^3; the small figure is called the **index** or **exponent** of r, and denotes the number of times that r enters as a factor into the given quantity.

N.B—When the co-efficient or the index of any quantity is *unity*, it is not written: *e.g.*, a, means $1 \times a^1$.

The Dimensions or Degree of an algebraic term means the number of separate letters contained in the term, if they were all written out in full, thus:—

a^4b^2c [$aaaabbc$] is of 7 dimensions or of the 7th degree. So $6a^2bc$ is of 4 dimensions or of the 4th degree, the numerical co-efficient 6 not being counted in speaking of dimensions or degree.

The Dimensions or Degree of a term is the sum of the indices of the several factors of the term.

The Dimensions or Degree of an Expression means the dimensions of the term of highest degree in that expression, *e.g.*, $3x^3 + 4x^2 + 7x + 6$ is an expression of three dimensions or of the 3rd degree, because $3x^3$, its term of highest degree, has 3 dimensions.

A Homogeneous expression *is one where each term has the same dimensions.* [Homogeneous from Gr. *homos* = the same, and *genos* = kind], *e.g.*, $3x^3 + 4x^2y + 2xy^2 + 7y^3$ is a *homogeneous expression of three dimensions.*

A positive term or quantity is one with a + sign before it.

A negative term or quantity is one with a − sign before it.

Like terms are such as have their *literal* or letter part exactly the same, *e.g.*, $6a$, $12a$, $24a$ are like terms; $7a^3$, $9a^3$, $12a^3$ are like terms; and $9a^2b$, $11a^2b$, $2a^2b$, are like terms: but, if the letter part differ at all, the terms are *unlike*; *e.g.*, $6a^2b$ and $6ab^2$ are unlike: $7a^2$ and $7a^3$ are unlike.

The arrangement of an algebraic expression should always be according to **ascending** or **descending** powers of some leading letter, or letter that occurs most frequently in the expression: *e.g.*, $9a^4 - 7a^3 + 6a^2 - 5a + 3$ is according to descending powers of a; while $1 + 3x + 6x^2 + 8x^3$ is according to ascending powers of x.

N.B.—The terms of an expression may be arranged in any order, provided they carry their signs along with them; thus, $5x^4 - 7x - 8x^2 + 9$ may be written $9 - 7x - 8x^2 + 5x^4$.

EXERCISES ON CHAPTER I.

The Use of Algebraic Signs.

Exercise 1.

Find the Arithmetical value of the following :—

A. **1.** $(3 + 9 - 2) \times (5 - 3)$.

2. $6 \times (3^2 + 4^3) \times (5^2 - 3^2)$.

3. $5 \times \sqrt[3]{8} \times \sqrt{3^2 + 4^2} - \sqrt{121}$.

4. $(\sqrt{13^2 - 5^2} + \sqrt[3]{9^2 - 17})^2$.

B. **1.** $(8 - 2 + 6) \times (7 - 4)$.

2. $8 \times (4 + 6^2) \times (4^3 - 5^2)$.

3. $(8^2 - \sqrt[3]{8}) \times (\sqrt[3]{27} + \sqrt{25})$.

4. $4\sqrt{4} + \sqrt{4 \times 4} + 9\sqrt{9} - \sqrt{9 \times 9}$.

C. **1.** $9 \times \sqrt{49} \times \sqrt[3]{7^2 - 22} - (6 + 3^3) \times \sqrt{4}$.

2. $\{(3 + 4 + 9) \times \sqrt[3]{8^2 + 9^2 - 20}\}^2$.

3. $(\sqrt{196} - \sqrt[3]{216})(4^2 + \sqrt{169})$.

4. $9\sqrt{9} - \sqrt{9 \times 9} + 25\sqrt{25} - \sqrt{25 \times 25}$.

Model Solution.

Find the Arithmetical value of

$$\{\sqrt[3]{14^2 + 5^3 + 5^2 - 3} \times \sqrt{6(3^3 - 4^2 - 5)}\}^2.$$

Given Expn. $= \{\sqrt[3]{196 + 125 + 25 - 3} \times \sqrt{6(27 - 16 - 5)}\}^2$.

$= \{\sqrt[3]{343} \times \sqrt{36}\}^2 = (7 \times 6)^2 = 42^2 = 1764$. **Ans.**

Exercise 2.

If $a = 1$, $b = 2$, $c = 4$, $d = 6$, $f = 0$, find the numerical value of the following :—

A. **1.** $6a + 3b - 2c + d - 4f$.

2. $7ab + 2bc - 4ad + bd$.

3. $c(a + 9) + d(b + 4) - a(f + 6)$.

4. $3(ad - ac) + 6(bd - af)$.

B. 1. $4b + 6c - 5d + 3f - a$.

2. $9ac - 4bd + 5ad - 3af$.

3. $2abc + 5bcd + 3acd - 2bdf$.

4. $bc\,(a + b) + cd\,(b + f) + ac\,(b + c)$.

C. 1. $d\,(7 + b) - c\,(a + 8) + b\,(f + 4)$.

2. $5\,(bc + ad) - 2\,(cd - ab)$.

3. $4cbd + 2dbf - 3cba + 5abd$.

4. $ab\,(d + f) + bd\,(c + a) - ac\,(b + a)$.

Model Solution.

If $a = 1$, $b = 2$, $c = 4$, $d = 6$, $f = 0$, find the numerical value of $abd\,(ab + bc) - bc\,(ab + bc + cf)$.

$$\begin{aligned}\text{Given Expn.} &= 1.2.6\,(1.2 + 2.4) - 2.4(1.2 + 2.4 + 4.0)\\ &= 12(2 + 8) - 8(2 + 8 + 0)\\ &= 12 \times 10 - 8 \times 10\\ &= 120 - 80 = 40.\quad \textbf{Ans.}\end{aligned}$$

Exercise 3.

If $a = 4$, $b = 25$, $c = 16$, $d = 0$, $x = 1$, $y = 9$, find the numerical value of :—

A. 1. $4\sqrt{b} - 2\sqrt{c} + 2\sqrt{bc}$.

2. $\sqrt{4a} - 5\sqrt{xy} + 2\sqrt{y}$.

3. $2\sqrt{4b + 11a} - 3\sqrt{c + y}$.

4. $\sqrt{4a^2 + 5c} - \sqrt{2cd} + \sqrt{y^2 + d^2}$.

B. 1. $3\sqrt{y} + 4\sqrt{c} - 3\sqrt{b}$.

2. $\sqrt{ca} + 6\sqrt{ay} - 4\sqrt{a}$.

3. $9\sqrt{3c + x} + 2\sqrt{2y - 2x}$.

4. $\sqrt{4ac} + \sqrt{bc + by} - 3\sqrt{a^2 + y}$.

C. 1. $2\sqrt{a}+3\sqrt{b}+4\sqrt{c}+2\sqrt{d}.$

2. $7\sqrt{4y}-2\sqrt{ab}+3\sqrt{c}.$

3. $\sqrt{9a+4c}+7\sqrt{7y+x}.$

4. $\sqrt{4x^2+2c}+6\sqrt{4y+ca}.$

Model Solution.

For the same values of a, b, c, d, x, y, find the value of

$$9\sqrt{3c+ay+a^2}+6\sqrt{abc+ady}.$$

$$\begin{aligned}\text{Given Expn.} &= 9\sqrt{3.16+4.9+4^2}+6\sqrt{4.25.16+4.0.9}.\\ &= 9\sqrt{48+36+16}+6\sqrt{1600+0}\\ &= 9\sqrt{100}+6\sqrt{1600}=9\times10+6\times40\\ &\qquad\qquad\qquad\qquad=90+240=330.\quad \textbf{Ans.}\end{aligned}$$

Exercise 4.

Find the Arithmetical values of the following if $x=2$, $y=3$, $z=4$, $p=5$, $n=6$, $r=1$:—

A. 1. $\dfrac{2p+3n}{y+z}+\dfrac{3x+y}{n+y}.$

2. $\dfrac{n^2-z^2}{(x+y)}+\dfrac{y^2-x^2}{(y-x)}.$

3. $\dfrac{(x+y+z)(x+y-z)}{x-r}+\dfrac{3xy}{n}$

4. $\dfrac{x+y+p}{\sqrt{(4n+r^2)}}+\dfrac{z+n-r}{\sqrt{(z^2-n-r)}}$

B. 1. $\dfrac{xy+yz}{p+r}+\dfrac{np+6x}{z+y}.$

2. $\dfrac{p^2+y^2}{z^2+r}+\dfrac{n^2+x^2}{p+y}.$

3. $\dfrac{(n+r-p)(n+r+y)}{z-x}+\dfrac{3xz}{nr}.$

4. $\dfrac{p+n+r}{\sqrt{3x+y^2+r}}+\dfrac{y+z-x}{\sqrt{4r+5y+n}}.$

C. 1. $\dfrac{nr+pz}{x+n+p}+\dfrac{py-yr}{x+y+r}.$

2. $\dfrac{z^2-x^2}{xy+n}-\dfrac{3p^2-2n^2}{2x-r}.$

3. $\dfrac{\sqrt{5y^2-p^2-x^2}}{x+2z-3y}+\dfrac{\sqrt{3n^2-11x^2}}{x+n-z}.$

4. $\dfrac{xy+pz+x+z}{\sqrt{2n^2-xz}}+\dfrac{xyz+p^2+r}{\sqrt{3z^2-p^2+x}}.$

Model Solution.

For the same *literal values* find the value of

$$\frac{\sqrt{3x^2+4y^2+r^2}}{x+2y-r}+\frac{\sqrt{8z^2-2p^2-2(y+z)}}{z+2p-n}.$$

$$\text{Given Expn.}=\frac{\sqrt{3.2^2+4.3^2+1^2}}{2+6-1}+\frac{\sqrt{8.4^2-2.5^2-2(3+4)}}{4+10-6}.$$

$$=\frac{\sqrt{12+36+1}}{7}+\frac{\sqrt{128-50-14}}{8}.$$

$$=\frac{\sqrt{49}}{7}+\frac{\sqrt{64}}{8}=\frac{7}{7}+\frac{8}{8}=1+1=2.\ \textbf{Ans.}$$

Exercise 5.

Express in Algebraical symbols :—

A. 1. The sum of z and y diminished by x.

2. Six times x added to seven times y and diminished by nine times z.

3. Three times a added to four times b is equal to five times c.

4. a times b is equal to x times y.

B. 1. The difference between x and y added to three times z.

2. Seven times a taken from six times b and added to three times c.

3. Twice x taken from five times y is equal to four times z.

4. p times m is equal to r times s.

C. 1. The product of a and x added to the product of b and y.

2. Four times f taken from nine times h and increased by three times d.

3. Ten times a taken from three times b is equal to twice c taken from d.

4. Three times the product of a, b, and c added to the double of d.

Model Solution.

Express symbolically

9 times the sum of $2m$ and n added to 6 times the product of $3p$ and $2q$.

$9(2m + n) + 6 \times 3p \times 2q$. **Ans.**

Exercise 6.

Write the following Algebraical expressions in words at full length :—

A. 1. $2a-3b+4c$.

2. $2x+y+pa$.

3. $2a-\frac{b}{2}+c$.

4. $3bc-4ay=5by$.

B. 1. $4p+2q-5r$.

2. $3y-4x-2z$.

3. $9a-2d+\frac{1}{2}x$.

4. $4by-2ax=0$.

C. 1. $c+4d-2b$.

2. $3y+5z-8x$.

3. $\frac{c}{2}+d-a=0$.

4. $5pq-2xy$.

Model Solution.

Express in words

$3abc-4(a+b)ab+6(a-c)ac$.

Three times the product of a, b, and c, diminished by 4 times the sum of a and b into their product, and increased by 6 times the difference between a and c into their product. **Ans.**

Exercise 7.

Use Algebraic symbols to answer the following :—

A. 1. I have b books in two cases ; and a books are in the smaller case ; how many have I in the larger one ?

2. Ten years ago I was x years old ; what will be my age ten years hence ?

3. A man earns x shillings per day, how many may he earn in a weeks?

4. Eight years ago A was four years older than B, who was y years old. What is A's age now?

B. 1. There are 20 boys in each class, and x classes in the school; how many boys are there in all in the school?

2. If n be an odd number, write down the next two odd numbers above and below n.

3. Two boys A and B have each x marbles at first. After playing for some time A wins y marbles. How many does each now have?

4. Three years ago I was m years old; how old shall I be seven years hence?

C. 1. There are 3 houses; each house has a windows, and each window b panes of glass. How many panes have they in all?

2. If x is an *even* number, write the four next *odd* numbers.

3. If A is p years old, in how many years will he be r years old?

4. From a yard measure I cut off three times in succession x inches, and then 6 inches. How many inches of the measure will now remain?

Model Solution.

If the 15th Regiment contained 80 more men, it would still number 40 less than the 4th Regiment, which numbers x men. How many men has the 15th Regiment?

No. of men in 15th Regiment is less than No. in 4th Regiment by 80 and 40 together, *i.e.*, is less than x by 120,

$\therefore$ No. of men in 15th Regiment $= x - 120$. **Ans.**

Chap. II—ADDITION.

§ 5. Addition in Algebra resembles compound addition in Arithmetic, like quantities being arranged in the same column. Attention has however to be paid to the **signs** in Algebra; if any term has no sign, + is understood.

The rule may be stated thus :—

Arrange all the given quantities, each with its own sign, so that like terms whether + or − fall in the same column; add all the + co-efficients and all the − co-efficients of the 1st column on the left; take the difference and set it down with its proper sign as the co-efficient of the first term of the required sum; do the same for each of the other columns in succession.

Note 1.—If the sum of all the + co-efficients just equals the sum of all the − co-efficients in any column, the sum of that column is zero, and there is no corresponding term in the answer.

Note 2.—The **Algebraic sum** of any number of quantities is distinguished from the Arithmetical sum by having regard to the *signs* of the separate quantities in Algebra.

Note 3.—It is not *always* necessary to arrange in columns; it saves time and labour to be able to add by **mentally** collecting like terms as they come *in line*, and doing the additions or subtractions of co-efficients mentally.

EXERCISES ON CHAP. II.—ADDITION.

Exercise 8.

Arrange in columns and add

A. 1. $3x + 5y + 8x + 6y + 7x + y$.
2. $7x - 6y + 2x - 3y + 5x - y$.
3. $-3a + 4b - 5b - 2a + 7b + 3a$.
4. $9c - 3d - 7c - 6d + 3c + 16d$.

B. 1. $4a + 3b + 5a + 9b + 7a + 3b$.
2. $7m - 2n + 8m - 3n + 2m - n$.
3. $-6x + 3y - 2x - 4y + 4x + 7y$.
4. $5a - 3b - 10a - 12b + 2a + 20b$.

C. 1. $2m + 3n + 8m + 6n + m + 2n$.
2. $5c - 3d + 6c - 2d + 7c - 4d$.
3. $-5x + 7y + 3x - 9y - 2x + 5y$.
4. $-7m + 2n - m - n + 10m - 16n$.

Model Solution.

Arrange and add : $14m - 8n + 6n - 7m + 3n - 2m$.

$$\begin{array}{r} 14m - 8n \\ -7m + 6n \\ -2m + 3n \\ \hline 5m + 3n \end{array} \quad \textbf{Ans.}$$

Exercise 9.

Collect and add the following, without arranging in columns.

A. 1. $2a - 3b + 5a - 6b + 4b - 9a + 6b$.
2. $5x + 9y - 3x - 6y + 2x - 5y - 3y$.
3. $5c - 9d + 7c + 3d - 8c - 12d + 4c$.
4. $3p + 2r - p - r + 6r - 8p - 3r$.

B. 1. $6x - 5y + 3y - 8x - 7x - 6y + 10x$.
2. $8p - 3s - 2p + 10p + 13s - 4p$.
3. $-9a + 2b + 5b + 7b + 4a - 6b - 2a$.
4. $5x + 4a - 9x - 7a + 11x - 15a - 5x$.

C. 1. $5p + 4c + 8p + 6c - 2p - c - 3p$.
2. $3a + 6b - 7a - 2b + a - 3b - 2a$.
3. $4x - 7y - 8x - 5y + 9y - 7x - 3x$.
4. $5x - 6y - 4y - 8x + 9y + 21x - y$.

Model Solution.

Collect and add the following :—

$19b - 7c + 8c - 20b + 6c + 11b - 4c$.

$$\begin{aligned}\text{Given Expression} &= (19b + 11b - 20b) + (8c + 6c - 7c - 4c)\\ &= (30b - 20b) + (14c - 11c)\\ &= 10b + 3c. \quad \textbf{Ans.}\end{aligned}$$

Exercise 10.

Add the following (i) by arranging in columns, and (ii) without so arranging :—

A. 1. $5a^2 - 6b^2 + 3c^2 - 9d^2 + 8b^2 - 4a^2 - 11d^2 + 6c^2 + 7a^2 - 3d^2 + 7b^2 - 16c^2 + 7d^2 - 9a^2 + 5c^2 - 2b^2$.
2. $8xy + 9yz - 7xz + 3xy + 4xz - 2yz - 5xy + 8yz - 4xz + 11yz - 2xy - 3xz$.
3. $14abc - 11bcd + 12acd - 3abc + 6acd + 5bcd - 9acd - 2abc - 7bcd + 10abc - 9bcd - 8acd$.
4. $4x^2y + 7xy^2 + 9x^2y^2 - 2xy^2 + 3x^2y - 5x^2y^2 - 6x^2y - 3xy^2 + 4x^2y^2 + x^2y - xy^2 - x^2y^2$.

B. 1. $9x^2 - 3y^2 + 8z^2 - 7x^2 - 6y^2 - 3z^2 + 4z^2 + 5y^2 - z^2 + 6x^2 + 5z^2 - 9y^2 - 11x^2 - 3y^2 - 5z^2$.
2. $3pq - 4pr + 6qr - 7ps + 8pr - 7pq - 3qr + 11ps - 5pq + 8ps - 13qr - 20pr + 4qr - 8pq - 17ps + qr$.

3. $11xyz + 6x^2y^2z^2 - 3xy^2z + 4xyz^2 - 5x^2yz$
$- 9x^2y^2z^2 - 7xyz + 8xy^2z - 13xyz^2 + 2xyz$
$- 9xy^2z + x^2y^2z^2 - 3x^2yz.$

4. $5abcd - 3ab^2c^2d + 11a^2bcd^2 + 4ab^2c^2d - 7abcd$
$- 9a^2bcd^2 + 8ab^2c^2d + 18abcd - 15a^2bcd^2$
$- 17ab^2c^2d - 11abcd + 12a^2bcd^2.$

C. 1. $6mnp + 3m^2n^2p^2 + 9m^3n^3p^3 - 11m^2n^2p^2$
$- 5m^3n^3p^3 - 8mnp + 11mnp - 17m^2n^2p^2$
$- 11m^3n^3p^3.$

2. $5abp - 7a^2b^2 - 3a^2q^2 + 4b^2q^2 - 11abp + 6a^2b^2$
$+ 9a^2q^2 - 7b^2q^2 + 11a^2q^2 - 6b^2q^2 - 2a^2b^2$
$+ 3abp.$

3. $2axyz + 3bx^2yz - 4cxy^2z^2 + 8axyz - 6bx^2yz$
$-11cxy^2z^2 - 5axyz + 7bx^2yz + 6cxy^2z^2 + 6axyz$
$+ 7cxy^2z^2 - 11bx^2yz.$

4. $3mnp^2 + 4m^2np - 7mn^2p + 10m^2n^2p^2 - 5m^2np$
$+ 6mn^2p - 9mnp^2 - 3m^2n^2p^2 + 8mn^2p - 3mnp^2$
$- 4m^2n^2p^2 - 6m^2np.$

Model Solution.

Add the following (i) by arranging in columns, and (ii) without so arranging :—

$10px^2z - 3qxz^2 - 11rxz - 7qxz^2 + 4rxz - 2px^2z + 8rxz - 5yx^2z$
$-17p^2xz - 12qxz^2 + 3rxz + 5qxz^2.$

(i) Arranging in columns.

$$\begin{array}{r}
10px^2z - 3qxz^2 - 11rxz. \\
-\ 2px^2z - 7qxz^2 + 4rxz. \\
-\ 5px^2z - 12qxz^2 + 8rxz. \\
-17px^3z + 5qxz^2 + 3rxz. \\
\hline
-14px^2z - 17qxz^2 + 4rxz. \\
\hline
\end{array}\quad \textbf{Ans.}$$

(ii) Without arranging in columns:—

$$\text{Given sum} = (10px^2z - 2px^2z - 5px^2z - 17px^2z) + (-3qxz^2 - 7qxz^2 - 12qxz^2 + 5pxz^3) + (-11rxz + 4rxz + 8rxz + 3rxz)$$

$$= (10px^2z - 24px^2z) + (-22qxz^2 + 5qxz^2) + (-11rxz + 15rxz)$$

$$= - 14px^2z - 17qxz^2 + 4rxz. \quad \textbf{Ans.}$$

Chap. III.—SUBTRACTION.

§ **6**. If b is to be subtracted from a, the result is expressed in Algebra by $a - b$; and if a is to be taken from b, the result is written $b - a$.

If $b + c$ is to be taken from a, the result is obviously $a - b - c$; for we have to take first b from a, which gives $a - b$; and then c from this, which gives $a - b - c$.

If $b - c$ is to be taken from a, the result is $a - b + c$; for if we take only b from a we get $a - b$; but if we subtract b we take away c too much; thus $a - b$ is c too small; hence $a - b + c$ is the final result.

If we take $2a - 3b + 4c$ from $5a - 7b + 6c$, we proceed very much as in compound subtraction in Arithmetic: and we get $3a - 4b + 2c$.

All these cases may be summarized in the following

Rule.—Change mentally the sign of each term of the subtrahend (or expression to be subtracted) and add.

EXERCISES ON CHAP. III.—SUBTRACTION.

Exercise 11.

A. 1. From $a - b$ take $a + b$.

2. From $3x - 4y - 5z$ take $2x - 2y - 3z$.

3. From a take $a - b - c$.

4. From $5x + 2y - 9z$ take $- x - y - z$.

B. 1. From $x + y$ take $x - y$.
2. From $2c - 5d - 3f$ take $c - 2d - f$.
3. From x take $x - y - z$.
4. From $3a + 2b - 4c$ take $-2a + 2b - 2c$.

C. 1. From c take $c + d$.
2. From $3p + 8q - 5r$ take $2p + 4q - 3r$.
3. From $5p + 4r - 3s$ take $-4p - 5r - 6s$.
4. From $p - 2q + 3s$ take $3p - q + s$.

Model Solution.

From $7x - 2y + 3z$ take $-2x + 4y - 3z$.

$$\begin{array}{r} 7x - 2y + 3z \\ -2x + 4y - 3z \\ \hline 9x - 6y + 6z \\ \hline \end{array} \quad \textbf{Ans.}$$

Exercise 12.

Subtract *twice* in succession:—

A. 1. $3xy - 7xz + 8yz$ from $9xy - 2xz + 7yz$.
2. $3p + 4q - r$ from $8p - q + 9r$.
3. $2a - 3b + 4c$ from $4a - 2b - 9c$.
4. $3pq - 2pr + 8qr$ from $11pq - 3pr - 5qr$.

B. 1. $5a - 7b - c$ from $12a - 9b - 4c$.
2. $3p + 2q - 4r$ from $9p - 5q - 6r$.
3. $4xy - 2xz + 3yz$ from $6xy + 7xz - 7yz$.
4. $xy - 3xz + 2yz$ from $4xy - 2xz + 3yz$.

C. 1. $2p - 3q + r$ from $5p - 2q - r$.
2. $2a - 7b + 3c$ from $11a - 5b + 7c$.
3. $3ab + 2ac - 5bc$ from $7ab - 9ac + 11bc$.
4. $2a^2 + 3b^2 - 2c^2$ from $7a^2 + 9b^2 - 5c^2$

Model Solution.

Subtract twice in succession :—

$5x^2 + 3y^2 - 2z^2$ from $7x^2 - 5y^2 + 6z^2$

$$\begin{array}{r} 7x^2 - 5y^2 + 6z^2 \\ 5x^2 + 3y^2 - 2z^2 \\ \hline 2x^2 - 8y^2 + 8z^2 \\ 5x^2 + 3y^2 - 2z^2 \\ \hline -3x^2 - 11y^2 + 10z^2 \\ \hline \end{array} \quad \textbf{Ans.}$$

Chap. IV.—EXTENDED MEANING OF THE SIGNS + AND —.

§7. The signs + and — denote *operations of an opposite kind*, viz., adding and taking away; hence they naturally come to be used to denote any operations or things which are opposite in character, or which are the reverse of one another. The following are important examples :—

(1) *Income* and *Expenditure* may be considered + and — additions to one's capital.

(2) *Buying* and *Selling* are positive and negative additions to one's stock.

(3) *Profit* and *Loss*, *Credit* and *Debit*, *Births* and *Deaths* are respectively positive and negative additions to income, or to bank account, or to population.

(4) Going in one way is a positive movement, and going in the opposite way is a negative movement.

Hence the signs + and — are often used advantageously in finding the *net result*, or *balance*, of a series of *transactions*, such as buying and selling, profit and loss, credit and debit.

Model Solution.

A librarian has in his library 386 books on English, 412 Novels, 133 Mathematical works and 249 works on History : he lends the first day 73 English, 93 Novels, 24 Mathematics, and 38 History ; next day he lends 46 English, 74 Novels, 19 Mathematics and 29 History ; and receives 57 English, 85 Novels, 31 Mathematics and 30 History ; on the third day he lends 59 English, 35 Novels, 12 Mathematics and 13 History, and receives 41 English, 23 Novels, 15 Mathematics and 16 History. What will be his stock of books in the library at the close of this third day?

English.	Novels.	Mathematics.	History.
+ 386	+ 412	+ 133	+ 249
− 73	− 93	− 24	− 38
− 46	− 74	− 19	− 29
+ 57	+ 85	+ 31	+ 30
− 59	− 35	− 12	− 13
+ 41	+ 23	+ 15	+ 16

(+ 484 − 178), (+ 520 − 202), (+ 179 − 55), (+ 295 − 80),
306 English, 318 Novels, 124 Mathematics, 215 History. **Ans.**

Exercise 13.

Employ the signs + and − to answer the following :—

A. 1. A dealer bought 6 cows, 8 sheep, and 3 horses ; then sold 5 sheep and 2 cows ; next bought 10 sheep and 5 horses : again bought 10 cows, 5 sheep, and 9 horses ; and finally sold 10 sheep, 6 cows, and 8 horses. What live stock has the dealer now on hand ?

2. A manufacturer hires 20 men, 30 women, and 12 boys on Monday : he sends away 10 men, 6 women, and 2 boys on Tuesday : and again 3 women and 4 boys on Wednesday : he takes on 17 more men, 5 more women, and 15 more boys on Thursday : lastly dismisses 5 men, 6 women, and 3 boys on Friday. How many employés does he now have at work ?

3. I have on hand 9 pounds, 11 shillings, and 10 pence ; I pay away 4 pounds, 3 shillings, and 2 pence ; I draw 20 pounds, 5 shillings, and one penny ; I next pay away 11 pounds, 10 shillings, and 7 pence ; I again pay away 1 pound, 1 shilling ; and lastly draw 13 pounds and 8 pence. What is now the state of my funds ?

B. 1. A furniture-hirer has in his shop 60 chairs, 20 cots, and 15 tables; he lends 18 chairs, 6 cots, and 3 tables to one person; and also 12 chairs, 4 cots, and 3 tables to another person; then he gets in 10 chairs, 3 cots, and 4 tables from a third customer; he lends 16 chairs, 7 cots, and 5 tables to a fourth party, and lastly gets in 5 chairs, 3 cots, and 7 tables. What furniture does he now have in his shop?

2. A bazaar man has in his store on Monday 500 measures of rice, 360 measures of ragi, and 400 measures of cholum; he sells that day 278 measures of rice, 200 of ragi, and 290 of cholum: on Tuesday he sells 140 measures of rice, 124 of ragi, and 90 of cholum: he replenishes his bazaar on Wednesday with 340 measures of rice, 250 of ragi, and 300 of cholum, selling the same day 130 measures of rice, 120 of ragi, and 190 of cholum: on Thursday he adds again to his store 220 measures of rice, 300 of ragi, and 380 of cholum: and sells the same day 280 measures of rice, 350 of ragi, and 420 of cholum. What has he in his store at the close of Thursday?

3. A general sets out on an expedition with 2000 infantry, 800 cavalry, and 480 artillery: in the first battle he loses 50 infantry, 20 cavalry, and 10 artillery; next followed a severe battle when he lost 840 infantry, 320 cavalry, and 160 artillery. Fresh supplies were however at hand consisting of 1200 infantry, 500 cavalry, and 320 artillery. Sickness carried off 110 infantry, 46 cavalry, and 28 artillery; and another battle cost him 280 infantry, 150 cavalry, and 62 artillery. What troops has the general now under his command?

C. 1. There are 3 classes in a school, and the numbers at opening are class I 24, class II 28, and class III 33 pupils. By the end of the 1st month there are withdrawn 4 from class I, 7 from class II, and 5 from class III; but there are enrolled as new pupils 7 in class I, 12 in class II, and 9 in class III: next month there are added 2 to class I, 5 to class II, and 3 to class III; while there are withdrawn 5 from class I, 6 from class II, and 7 from class III. What are the numbers now on the roll?

2. A gentleman has 3 bags; one contains 100 sovereigns, another 80 crowns, and the third 70 shillings. He puts in 16 sovereigns, and takes out 24 crowns, and puts in 12 shillings;

next he takes out 39 sovereigns, puts in 17 crowns, and takes out 30 shillings: thereafter he puts in 12 sovereigns, takes out 30 crowns, and puts in 18 shillings; lastly he takes out 25 sovereigns, 14 crowns, and 39 shillings. What has he now in the bags?

3. A Railway train starts from a certain station with 20 first class, 54 second class, and 568 third class passengers. At the second station, 2 first class passengers enter and 5 leave; 6 second class enter and 2 leave; and 32 third class enter and 10 leave. At the 3rd station, 4 first class enter and 5 leave; 8 second class enter and 12 leave; and 14 third class enter and 18 leave. At the fourth station, 2 first class enter and 7 leave; 5 second class enter and 11 leave; and 36 third class enter and 47 leave. What passengers are now in the train?

Chap. V.—THE USE OF BRACKETS.

§8. A pair of brackets is used in algebra to mean that all the terms within them are to be considered a single quantity; and their use is meant to shorten algebraic expressions. Just as in ordinary English we abbreviate the expression *a red colour, a blue colour, and a green colour,* into a red, a blue, and a green colour, so in algebra we abbreviate $rc + bc + gc$ into $(r + b + g)c$.

When there is only **one pair of brackets** in an algebraical expression, and we wish to remove the brackets for the purpose of simplifying the whole, the following points must be carefully attended to:—

(i) If + be the sign before a bracketed expression, the brackets may be removed without affecting the sign of any term within.

(ii) If — be the sign before a bracketed expression, we must change the sign of *each term* within, if we remove the brackets.

(iii) If a co-efficient be prefixed to a bracketed expression, it must be multiplied into *each term* of the quantity if we remove the brackets; and if the — sign be before the co-efficient, it will also change the sign of each term of the product.

Examples—$3(a^2-ab)-2a(b-a)=3a^2-3ab-2ab+2a^2$
$$=5a^2-5ab.$$

§9. An expression often has **two or more pairs of brackets**; the square brackets [ ] are used as the outside or most comprehensive brackets; within these may be one or more pairs of face brackets { }; and within each pair of face brackets may be one or more pairs of common brackets ( ); and within each pair of common brackets may be one or more *vinculums* $\overline{\qquad\qquad}$.

In order to simplify such a complex expression, we apply the above rules for simple brackets first to the *innermost* bracketed portions, removing the brackets one pair at a time. It is not *necessary* to proceed in this way; but it is usually more convenient. With practice the process may be shortened, and two or more pairs of brackets removed at the same time. [See *Model Solution* to Exer. 15.]

§10. For the **Insertion of Brackets** we must use the converse of the above rules, as follows:—

(i) We may put any number of terms within brackets and prefix the sign + without any other change, *e.g.*, $a+2b-3c=a+(2b-3c)$.

(ii) If we prefix the sign —, we must reverse the signs of all terms within the brackets; *e.g.*, $a-2b+3c=a-(2b-3c)$.

(iii) If any *factor* is common to two or more terms of an expression we may place it outside brackets and remove it from each term;

Example—$3a^2b-3ab^2=3ab\times(a-b)$.

Exercise 14.

Remove the brackets from the following expressions and simplify them :—

A. 1. $2x + 3y + z - (x - y - z)$.
2. $3a - 2b + 3c - (2a - 2b - 2c)$.
3. $5x - 3\{3x - (4x - y) + 2y)\}$.
4. $3n - 4m - 5\{2n - 3(m - n)\}$.

B. 1 $2a - b + 5c - (a - 2b - 3c)$.
2. $4m - 5n + 6p - (3m - 2n - 4p)$.
3. $12a + 7b - 4\{3b - (5a - 9b)\}$.
4. $6a + 11b - 3\{4b - 3a - (2a - b)\}$.

C. 1. $3c + 2d + 7e - (2c - 3d - 4e)$.
2. $4x + 7y - 9z - (3x + 5y - 2z)$.
3. $10a + 5b - 6\{3a - (6a - 3b)\}$.
4. $9x + 5y - 2\{3y - 6x - (3x - 2y)\}$.

Model Solution.

Remove brackets and simplify $9m - 7n - \{8n - 4m - (3n - 5m)\}$

Given Expression $= 9m - 7n - \{8n - 4m - 3n + 5m\}$

$= 9m - 7n - \{m + 5n\}$

$= 9m - 7n - m - 5n.$

$= 8m - 12n.$ **Ans.**

Exercise 15.

Remove brackets and simplify :—

A. 1. $8 - [3 - \{6 - (4 - \overline{3 - 7})\}]$.
2. $3x - [6x - \{5x - (2x - \overline{7x - 3x})\}]$.

3. $6x - 9y - [3x + 8y - \{5x - 11y - (4x - 6y - \overline{7x - 12y})\}]$.

4. $8m + 6n - [5n - 3m - \{4m - (3n - 2m) - (6n - 7m)\} - 4\{6n - (4m - 3n) - (11m - 4n)\}]$.

B. 1. $11 - [6 - \{8 - (5 - \overline{12 - 4})\}]$.

2. $6a - [8a - \{3a - (7a - \overline{11a - 6a})\}]$.

3. $9a - 7b - [5a + 6b - \{3a - 4b - (6a - 8b - \overline{10a - 7b})\}]$.

4. $11x + 8y - [6x + 5y - \{3x + 4y - (2x - 5y) - (4x - 3y)\} - 3\{8x - 11y - (14x - 6y) - (9x - 4y)\}]$.

C. 1. $9 - [6 - \{5 - (3 - \overline{8 - 4})\}]$.

2. $5m - [9m - \{6m - (3m - \overline{8m - 2m})\}]$.

3. $8m - 6n - [4m - 2n - \{7m + 5n - (11m - 4n - \overline{6m + 4n})\}]$.

4. $7a + 5b - [3a + 2b - \{5a - (8a - 4b) - (4a - 2b)\} - 5\{3b - (2a - 5b) - (6a - 9b)\}]$.

Model Solution.

Remove brackets from and simplify :—

$9m + 7n - [6n + 3m - \{6m - (3m + 7n) - (4n - 8m)\} - 3\{9n - (8n - 3m) - (5m - 2n)\}]$.

Given Expn.

$= 9m + 7n - [6n + 3m - \{6m - 3m + 7n - 4n + 8m\} - 3\{9n - 8n + 3m - 5m + 2n\}]$

$= 9m + 7n - [6n + 3m - \{11m + 3n\} - 3\{-2m + 3n\}]$

$= 9m + 7n - [6n + 3m - 11m - 3n + 6m - 9n]$

$= 9m + 7n - [-2m - 6n]$

$= 9m + 7n + 2m + 6n$

$= 11m + 13n$. **Ans.**

Chap. VI.—MULTIPLICATION.

§11. Multiplication of Monomials.

Under **definitions** it has been explained that the product of *unlike quantities* such as a and b is denoted in algebra by writing them together without any signs, thus ab. Also it follows from the definition of *power* or *index* that, when *like quantities* are to be multiplied, we simply *add* the indices: for $a^3 = aaa$, and $a^5 = aaaaa$; hence $a^3 \times a^5 = aaa \times aaaaa = a^8$ (by def.).

Numerical Co-efficients are multiplied as in arithmetic:

Example.—$7a^2b^3 \times 6a^3b^3 = 42a^5b^6$.

Literal Co-efficients cannot be combined in this way except when they are the same letter. Thus $7ax^2 \times 3ax^3 = 21a^2x^5$; but $7ax^2 \times 3bx^3 = 21abx^5$.

For Signs, the product of two quantities with *like* signs is +, but with *unlike* signs is −.

$$\text{Thus} -2x^2 \times (-3x^3) = +6x^5;$$
$$\text{but} +2x^2 \times (-3x^3) = -6x^5.$$

The reason of the fact that — into — gives + is simple. The sign — may be called the **symbol of reversal;** now if a thing or action is *twice reversed*, it is of necessity restored to the original or unreversed condition.

Obs. $m + m + m + m = 4m$;
but $m \times m \times m \times m = m^4$.

Exercise 16.

Multiply—

A. 1. $3ax^2$ by $2bx^3$; $4x^2y^3$ by $-3x^3y^4$; and $-6a^2bx^2y$ by $-4a^2b^3x^2y^4$.

2. $4pq^2 - 3p^2q + 5pq$ by $6p^3q^3$.

3. $3a^2bc - 2ab^2c - 4abc^2 - 6a^2b^2c^2$ by $-3abc$.

4. $-4xyz^2 - 3xy^2z + 7x^2yz - 18xyz$ by $6x^3yz^3$.

B. 1. $4p^2x^2$ by $3px^4$; $2x^4y^2$ by $-x^3y^4$; and $-8a^3bx^2$ by $-5a^2b^3x^3$.

2. $2bx^2 - 3c^2y + 4az^3$ by $-3a^2bc^2xy^2z$.

3. $5a^2xy^2 - 3ax^2y - 2axy + 4a^2x^2y^2$ by $-7a^3x^2y$.

4. $-2pq^2x^3z + 5p^2qxy^2 - 3p^2q^2x^2z^2 - pqz^3y^3$ by $4p^3qxy^2z$.

C. 1. $5xy^2$ by $9x^3y^4$; $6p^2x^3y$ by $-2px^2y^3$; and $-7a^2bxy^4$ by $-4a^3b^2x^2y^2$.

2. $6ax^2 - 7b^2y^3 - 3c^3z^2$ by $-2abx^2yz^3$.

3. $2pq^2x - 3p^2qx^2 - 4p^3q^2x^3$ by $5p^2qx^2$.

4. $4a^2bxy^2 - 5ab^2x^3y - 2abx^3y^2 - 3a^2b^2x^2y^3$ by $-6a^3b^3x^4y^2$.

Model Solution.

Multiply: $5a^3bx^2y^2 - 3ab^3xy^4 + 7a^2b^2x^3y^5$ by $-4a^4b^4x^3y^5$.

$$\begin{array}{l} 5a^3bx^2y^2 - 3ab^3xy^4 + 7a^2b^2x^3y^5 \\ \underline{-4a^4b^4x^3y^5.} \\ \underline{-20a^7b^5x^5y^7 + 12a^5b^7x^4y^9 - 28a^6b^6x^6y^{10}.} \quad \textbf{Ans.} \end{array}$$

§ 12. Multiplication of Polynomials.

If the multiplier and multiplicand each consist of several terms, then the product is the sum of the separate products of each term of the multiplicand by each term of the multiplier. Thus $p(a - b + c) = pa - pb + pc$;

and $(p-q)(a-b+c) = (p-q)a - (p-q)b + (p-q)c$

$$= pa - qa - pb + qb + pc - qc.$$

The **General Rule for Multiplication** may be summed up as follows:—

Arrange multiplicand and multiplier both according to ascending, or both according to descending, powers of the leading *letter; place the multiplier at the* **left-hand** *under the multiplicand, and work from left to right; now multiply each term of the multiplicand in succession by each term of the multiplier according to the rules given above for monomials; arrange like quantities in the same vertical columns and add.*

Exercise 17.

Multiply—

A. 1. $2x + 3$ by $3x + 7$: and $4x + 6$ by $5x - 3$.

2. $4a^2 - 3a + 5$ by $2a + 3$.

3. $3x^2 - 2x + 9$ by $5x + 4$.

4. $x^3 - 2x^2 + 3x - 1$ by $2x - 4$.

B. 1. $3x + 5$ by $4x + 6$: and $7x + 3$ by $2x - 4$.

✓ 2. $5y^2 - 2y + 7$ by $3y - 2$.

✓ 3. $7y^2 - 6y + 8$ by $2y + 3$.

4. $2p^3 - 3p^2 + 2p - 1$ by $4p - 6$.

C. 1. $3y + 8$ by $2y + 5$: and $4y + 6$ by $5y - 7$.

2. $2x^2 - 2x + 9$ by $4x - 3$.

3. $2p^2 - 8p + 7$ by $6p + 4$.

4. $3a^3 - 4a^2 - 5a + 2$ by $2a - 7$.

Model Solution.

Multiply $5x^3 - 3x^2 + 6x - 7$ by $3x + 8$

$$\begin{array}{rrrrr}
5x^3 & -\,3x^2 & +\,6x & -\,7 & \\
3x & +\,8 & & & \\
\hline
15x^4 & -\,9x^3 & +\,18x^2 & -\,21x & \\
 & +\,40x^3 & -\,24x^2 & +\,48x & -\,56 \\
\hline
15x^4 & +\,31x^3 & -\,6x^2 & +\,27x & -\,56.
\end{array}$$

Ans.

Exercise 18.

Multiply—

A. 1. $x^2 + x - 1$ by $x^2 - x + 1$.

✓ 2. $6x^2 - 7x + 8$ by $3x^2 - 4x + 5$.

3. $y^3 - 3y + 2$ by $y^3 - 3y^2 + 2$.

4. $x^2 + 4xy + 5y^2$ by $x^2 - 3xy - 2y^2$.

B. 1. $p + 2q - r$ by $p - q + 3r$.

✓ 2. $3a + 2b + c$ by $a + 2b + 3c$.

3. $m^3 - 3m^2 + 3m - 1$ by $m^2 + 3m + 1$.

4. $2x^2 - 3xy + 4y^2$ by $3x^2 + 2xy - y^2$.

C. 1. $2x^3 - 3x^2 + 2x$ by $2x^2 + 3x + 2$.

2. $3p^2 - p + 2$ by $1 + 5p + p^2$.

3. $a^3 - 2a^2b + 3ab^2 + 4b^3$ by $a^2 - 2ab - 3b^2$.

4. $6y^2 - 7y + 2$ by $y^2 + 2y + 3$.

Model Solution.

Multiply $7x^2 - 9xy + 10y^2$ by $3x^2 + 2xy - 3y^2$

$$\begin{array}{lllll}
7x^2 & -9xy & +10y^2 & & \\
3x^2 & +2xy & -3y^2 & & \\
\hline
21x^4 & -27x^3y & +30x^2y^2 & & \\
 & +14x^3y & -18x^2y^2 & +20xy^3 & \\
 & & -21x^2y^2 & +27xy^3 & -30y^4 \\
\hline
21x^4 & -13x^3y & -9x^2y^2 & +47xy^3 & -30y^4. \\
\hline
\end{array}$$

Ans.

Exercise 19.

Multiply—

A. 1. $a^2 + 4a + 15$ by $a^3 - 4a + 1$.

2. $6y^2 - 7y + 2$ by $y^2 + 2y + 3$.

3. $3y^3 + 7y^2 - 5y + 1$ by $2y^2 - 3y + 2$.

4. $a^3 + a^2b + ab^2 + b^3$ by $a^2 - ab + b^2$.

B. 1. $3a^2 - 2ab + 4b^2$ by $a^2 - 7ab + 3b^2$.

2. $1 + 2y + 4y^2 + 8y^3$ by $1 + 4y + 5y^2$.

3. $8x^3 - 4x^2y + 7xy^2 + 9y^3$ by $2x^2 + 3xy - 4y^2$.

4. $y^3 + ay^2 - by + c$ by $y^2 - ay - b$.

C. 1. $2y^3 - 5y^2 + 4y - 1$ by $3y^2 - y + 2$.

2. $9 + 7a + 5a^2 + 3a^3$ by $4 + 5a - 6a^2$.

3. $9x^4 - 7x^3 + 5x^2 - 3x + 4$ by $3x^3 + 8x^2 + 5x - 6$.

4. $z^3 - az^2 + bz - c$ by $z^2 - az + b$.

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Model Solution.

$$
\begin{array}{l}
3x^3 - 4x^2y + 5xy^2 - y^3 \\
\underline{2x^2 + 3xy - 4y^2} \\
6x^5 - 8x^4y + 10x^3y^2 - 2x^2y^3 \\
\qquad + 9x^4y - 12x^3y^2 + 15x^2y^3 - 3xy^4 \\
\qquad\qquad\quad - 12x^3y^2 + 26x^2y^3 - 20xy^4 + 4y^5 \\
\hline
6x^5 + x^4y - 14x^3y^2 + 29x^2y^3 - 23xy^4 + 4y^5 \quad \textbf{Ans.}
\end{array}
$$

§ 13. Product of Homogeneous Expressions.

If multiplicand and multiplier be each a homogeneous expression, their product will also be homogeneous, and its dimensions will be equal to the sum of the dimensions of the multiplicand and multiplier.

There is no more important law in all Algebra than this **Law of Homogeneity**. For example, $(a^2 + 2ab + b^2) \times (a^3 + a^2b + ab^2 + b^3) = a^5 + 3a^4b + 4a^3b^2 + 4a^2b^3 + 3ab^4 + b^5$. Here the multiplier is a homogeneous expression of two dimensions, the multiplicand one of three dimensions; and the product one of five dimensions. It is clear that this *must* be the case, because, at each little step of the process, we multiply a term of the second degree by one of the third degree; hence each of the partial products is a term of the fifth degree.

As the terms will be homogeneous for each step in the process, we have a ready check on the accuracy of our work by noting this carefully.

Exercise 20.

Find the continued product of the following:—

A. 1. $(a+b)(a+c)(a+d)$.
2. $(3x+4y)(2x+6y)(5x+8y)$.
3. $(9a-3b)(4a+7b)(3a-5b)(2a-4b)$.
4. $(a^2+ax+x^2)(a^2-ax+x^2)(a-x)$.

B. 1. $(x+y)(x+z)(x+p)$.
2. $(4a+6c)(5a+7c)(6a+2c)$.
3. $(8x+6y)(9x-3y)(2x-4y)(3x+5y)$.
4. $(ax+cy)(ac+xy)(ax-cy)(ac-xy)$.

C. 1. $(m+n)(m+p)(m+q)$.
2. $(6x-3y)(5x-4y)(3x-7y)$.
3. $(3a-4b)(5a+6b)(2a-3b)(4a-3b)$.
4. $(3+x)(4+x)(5-x)(6-x)$.

Model Solution.

Find the continued product of—

$$(5x-6y)(4x-3y)(8x-2y)(4x-5y).$$

1st. Product of $5x-6y$ & $4x-3y$. 2nd. Product of $8x-2y$ & $4x-5y$.

$$\begin{array}{lll} 5x-6y & & \\ 4x-3y & & \\ \hline 20x^2-24xy & & \\ \quad\;\; -15xy+18y^2 & & \\ \hline 20x^2-39xy+18y^2 & & \\ \hline \end{array} \qquad \begin{array}{lll} 8x-2y & & \\ 4x-5y & & \\ \hline 32x^2-8xy & & \\ \quad\;\; -40xy+10y^2 & & \\ \hline 32x^2-48xy+10y^2 & & \\ \hline \end{array}$$

Now multiply these two results together.

$$\begin{array}{l}
20x^2 - 39xy + 18y^2 \\
32x^2 - 48xy + 10y^2 \\
\hline
640x^4 - 1248x^3y + 576x^2y^2 \\
\qquad\quad - 960x^3y + 1872x^2y^2 - 864xy^3 \\
\qquad\qquad\qquad\qquad + 200x^2y^2 - 390xy^3 + 180y^4 \\
\hline
640x^4 - 2208x^3y + 2648x^2y^2 - 1254xy^3 + 180y^4 \quad \text{Ans.}
\end{array}$$

§ 14. Multiplication with fractional co-efficients.

If the co-efficients of any terms are fractional, we use the ordinary process: and multiply the fractional co-efficients by the rules of Arithmetic.

It is to be noted that the order of multiplication is of no moment: hence, if need be, we may turn multiplicand into multiplier in any case.

Exercise 21.

Multiply—

A. 1. $\frac{1}{3}x^2 - \frac{1}{2}xy + \frac{2}{3}y^2$ by $\frac{1}{2}x + \frac{1}{3}y$.
2. $\frac{1}{2}p^2 + \frac{1}{3}p + \frac{1}{4}$ by $\frac{1}{2}p - \frac{1}{3}$.
3. $\frac{2}{3}m^2 - \frac{1}{4}mn + \frac{1}{2}n^2$ by $\frac{1}{2}m - \frac{1}{3}n$.
4. $\frac{3}{4}a^2 - \frac{1}{2}ax + \frac{2}{3}x^2$ by $\frac{2}{3}a + \frac{1}{2}x$.

B. 1. $\frac{2}{3}a^2 - \frac{1}{4}ab + \frac{1}{2}b^2$ by $\frac{5}{3}a - \frac{5}{2}b$.
2. $\frac{1}{4}y^2 - \frac{3}{4}xy + \frac{1}{2}x^2$ by $\frac{2}{3}y - \frac{3}{2}x$.
3. $\frac{2}{3}p^2 - \frac{1}{6}pq + \frac{1}{2}q^2$ by $\frac{1}{2}p - \frac{1}{3}q$.
4. $\frac{1}{2}x^2 - \frac{1}{8}x + \frac{1}{4}$ by $\frac{1}{4}x - \frac{1}{2}$.

C. 1. $\frac{2}{3}m^2 - \frac{1}{4}mn + \frac{1}{2}n^2$ by $\frac{1}{3}m - \frac{2}{3}n$.
2. $\frac{1}{4}x^2 - \frac{3}{2}xy + \frac{1}{2}y^2$ by $\frac{2}{3}x - \frac{3}{2}y$.
3. $\frac{5}{4}a^2 + \frac{3}{4}ab - \frac{1}{4}b^2$ by $\frac{1}{2}a - \frac{1}{3}b$.
4. $\frac{1}{2}p^2 - \frac{2}{3}pq + \frac{1}{3}q^2$ by $\frac{3}{4}p - \frac{3}{2}q$.

Model Solution.

Multiply $\frac{2}{3}a^2 - \frac{1}{4}ab + \frac{1}{3}b^2$ by $\frac{1}{4}a - \frac{3}{4}b$

$$\begin{array}{l}
\frac{2}{3}a^2 - \frac{1}{4}ab + \frac{1}{3}b^2 \\
\frac{1}{4}a \quad - \frac{3}{4}b \\
\hline
\frac{1}{6}a^3 - \frac{1}{16}a^2b + \frac{1}{12}ab^2 \\
\qquad - \frac{1}{2}a^2b + \frac{3}{16}ab^2 - \frac{1}{4}b^2 \\
\hline
\frac{1}{6}a^3 - \frac{9}{16}a^2b + \frac{13}{48}ab^2 - \frac{1}{4}b^2 \quad \textbf{Ans.} \\
\hline
\end{array}$$

Chap. VII.—DIVISION.

§ **15**. As this is the converse operation to Multiplication, all the methods for Division will be the converse of those for Multiplication.

The process for Monomials. Divide numerical co-efficients as in arithmetic; and, to divide like literal quantities, subtract the index of the divisor from that of the dividend; the sign of the quotient is + or — according as signs of the terms are like or unlike. *Examples*

$12x^7 \div 3x^3 = 4x^4$; $15a^3b^2c \div (-5abc) = 3a^2b$.

§ **16**. **Division of Polynomials**. *The method may be summarised thus:—Arrange the divisor and dividend each according to ascending, or descending, powers of some leading letter; then divide the first (that is left-hand) term of the dividend by the first term of the divisor, and set this down as the first term of the quotient; multiply each term of the divisor by this, set down the product under the dividend and subtract as in arithmetic; now take down one or more terms of the dividend as may be necessary, and proceed in this way till all the terms of the dividend have in succession been taken down.*

§ 17. **The Law of Homogeneity.** This is of equal importance in division as in multiplication. It is as follows :—**If dividend and divisor be each homogeneous, so is the quotient : and its dimensions = the difference between the dimensions of dividend and divisor.**

Example : $(a^4 - 2a^2b^2 + b^4) \div (a - b) = a^3 + a^2b - ab^2 - b^3$, that is, a homogeneous quantity of 4th degree divided by one of the first degree gives a homogeneous quotient of the 3rd degree.

§ 18. **Arrangement in Division.** When we have a long divisor and dividend it is better to put the divisor on the *right hand*, (and not on the left as in arithmetic,) with the quotient under the divisor, separated from it by a line. There is less risk of making a slip of sign, &c., in carrying the eye from the new term of the quotient to each term of the divisor, as we perform the steps of multiplication.

The arrangement takes this shape :—

$$\textbf{Dividend}\left|\begin{array}{l}\textbf{Divisor}\\ \hline \textbf{Quotient}\end{array}\right.$$

Exercise 22.

Divide—

A. 1. $54a^3x^5$ by $18ax^3$; and $34ab^5y^4$ by $-17b^2y^2$.

2. $15x^7 - 20x^6 + 10x^5$ by $5x^3$.

3. $6x^3 + 19x^2 - 2x - 30$ by $2x + 5$.

4. $12a^2x + 24a^2y + 28a^2z + 9abx + 18aby + 21abz$ by $4a + 3b$.

B. 1. $60ac^3x^9$ by $12ca^6$; and $-42b^4c^2y^9$ by $14bcy^5$.

2. $16y^7 - 24y^6 - 28y^5$ by $4y^4$.

3. $21y^4 - 40y^3 + 43y^2 - 36y$ by $3y - 4$.

4 $8a^3x^3 + 24a^2x^2y + 24axy^2 + 8y^3$ by $2ax + 2y$.

C. 1. $21a^7b^2x^9$ by $7a^3x^6$; and $-12a^6b^2x^8$ by $6a^4bx^5$.

2. $48x^6 - 42x^5 + 36x^4$ by $6x^3$.

3. $12m^4 - 29m^3 + 27m^2 - 20m$ by $3m - 5$.

4. $6a^3x^3 - 18a^2x^2z + 18axz^2 - 6z^3$ by $3a - 3zx$.

Model Solutions.

1. $27a^7b^3x^7 \div 3a^4bx^3 = 9a^3b^2x^4$.

2. $-12x^5 + 20x^4 - 24x^3 + 36x^2 \div -4x^2 = 3x^3 - 5x^2 + 6x - 9$.

3. $28x^5 - 69x^4 + 67x^3 - 68x^2 + 21x \div 4x - 7$.

$$
\begin{array}{llll}
28x^5 - 69x^4 + 67x^3 - 68x^2 + 21x & \big(& 4x - 7 \\
 & & \overline{7x^4 - 5x^3 + 8x^2 - 3x.} \\
\underline{28x^5 - 49x^4} \\
\quad -20x^4 + 67x^3 \\
\quad \underline{-20x^4 + 35x^3} \\
\qquad\quad 32x^3 - 68x^2 \\
\qquad\quad \underline{32x^3 - 56x^2} \\
\qquad\qquad -12x^2 + 21x \\
\qquad\qquad \underline{-12x^2 + 21x}
\end{array}
$$

Ans. $7x^4 - 5x^3 + 8x^2 - 3x$.

Exercise 23.

Divide—

A. 1. $x^5 + 4x^4 + 7x^3 + 7x^2 + 4x + 1$ by $x^2 + 2x + 1$.

2. $x^5 - x^4y - xy^4 + y^5$ by $x^2 - 2xy + y^2$.

3. $6x^7 - 17x^6 + 34x^5 - 40x^4 + 20x^3 + 3x^2 - 24x + 18$ by $2x^3 - 3x^2 + 4x - 3$.

4. $\frac{1}{4}x^4 - \frac{43}{36}x^2 + \frac{9}{16}$ by $\frac{1}{2}x^2 + \frac{2}{3}x - \frac{3}{4}$.

B. 1. $6x^5 + 5x^4 - 25x^3 + 31x^2 - 13x + 2$ by $2x^2 - 3x + 2$.

2. $a^5 + 3a^4b + 4a^3b^2 + 4a^2b^3 + 3ab^4 + b^5$ by $a^2 + 2ab + b^2$.

3. $12x^8 - 8x^7 + 6x^6 + x^5 - 52x^4 + 61x^3 - 34x^2 + 14x - 4$ by $3x^3 - 2x^2 + 6x - 4$.

4. $\frac{1}{4}a^4 + x^4$ by $\frac{3}{4}a^2 - \frac{3}{2}ax + \frac{3}{2}x^2$.

C. 1. $3x^4 - 23x^3y + 27x^2y^2 - 34xy^3 + 12y^4$ by $x^2 - 7xy + 3y^2$.

2. $p^5 + 2p^4q + 3p^3q^2 + 3p^2q^3 + 2pq^4 + q^5$ by $p^2 + pq + q^2$.

3. $15x^8 - 44x^7 + 72x^6 - 87x^5 + 42x^4 + 27x^3 - 87x^2 + 70x - 36$ by $3x^3 - 7x^2 + 6x - 4$.

4. $\frac{9}{8}x^4 - \frac{3}{2}ax^3 + \frac{1}{2}a^2x^2 - \frac{2}{9}a^4$ by $\frac{3}{4}x^2 - \frac{1}{2}ax + \frac{1}{3}a^2$.

Model Solution.

Divide $\frac{3}{4}x^5 - 4x^4 + \frac{77}{8}x^3 - \frac{43}{4}x^2 - \frac{33}{4}x + 27$ by $\frac{1}{2}x^2 - x + 3$.

$$\begin{array}{l}
\frac{3}{4}x^5 - 4x^4 + \frac{77}{8}x^3 - \frac{43}{4}x^2 - \frac{33}{4}x + 27 \left(\begin{array}{l} \frac{1}{2}x^2 - x + 3 \\ \hline \frac{3}{2}x^3 - 5x^2 + \frac{1}{4}x + 9 \end{array}\right. \\
\underline{\frac{3}{4}x^5 - \frac{3}{2}x^4 + \frac{9}{2}x^3} \\
\qquad -\frac{5}{2}x^4 + \frac{41}{8}x^3 - \frac{43}{4}x^2 \\
\qquad \underline{-\frac{5}{2}x^4 + 5x^3 - 15x^2} \\
\qquad\qquad \frac{1}{8}x^3 + \frac{17}{4}x^2 - \frac{33}{4}x \\
\qquad\qquad \underline{\frac{1}{8}x^3 - \frac{1}{4}x^2 + \frac{3}{4}x} \\
\qquad\qquad\qquad \frac{18}{4}x^2 - \frac{36}{4}x + 27 \\
\qquad\qquad\qquad \underline{\frac{9}{2}x^2 - 9x + 27}
\end{array}$$

Ans. $\frac{3}{2}x^3 - 5x^2 + \frac{1}{4}x + 9$.

Exercise 24.

Find the remainder after dividing—

A. 1. $7x^3 + 9 + 96x^2 - 27x$ by $7x - 2$.

2. $16 + 9a^2 - 35c^2 + 7ac$ by $3a + 7c$.

3. $2a^3 + 12 - 14a - 38a^2 + 15a^4$ by $2 - 4a - 5a^2$.

4. $\frac{1}{6}a^4 - \frac{13}{6}a^3x - \frac{5}{2}ax^3 + \frac{1}{2}x^4 + \frac{19}{3}a^2x^2$ by $\frac{1}{2}a^2 - 2ax + \frac{2}{3}x^2$.

B. 1. $10 - 13x^2 + 100x^3 - x$ by $3 + 25x$.

2. $12a^2 + 13 - 12ac - 36c^2$ by $4a - 9c$.

3. $9x^3 - 6x^2 - 2x + 4 + x^5 - 5x^4$ by $x^2 - 3x + 2$.

4. $\frac{9}{16}a^4 + \frac{5}{3} + a - \frac{7}{4}a^2 - \frac{3}{4}a^3$ by $\frac{3}{8}a^2 - \frac{1}{4}a - \frac{2}{3}$.

C. 1. $8 - 2xy - 15y^2 + 96x^2$ by $12x - 5y$.
2. $36 + 6x^2 - 30x$ by $2x - 7$.
3. $3x^3 - 2x + 3x^2 - 2 - 4x^4 + x^5$ by $x^2 - x - 2$.
4. $36x^2 - 4xy + \frac{1}{2} - 5xy + \frac{1}{9}y^2$ by $6x - \frac{1}{3}y - \frac{1}{2}$.

Model Solution.

Find the remainder after dividing $\frac{1}{4}x^3 + \frac{1}{36}xy^2 + \frac{1}{6}y^3$ by $\frac{1}{2}x + \frac{1}{2}y$.

$$\begin{array}{l}
\frac{1}{8}x^3 + \frac{1}{86}xy^2 + \frac{1}{6}y^3 \;\Big(\begin{array}{l}\frac{1}{2}x + \frac{1}{3}y \\ \hline \frac{1}{2}x^2 - \frac{1}{3}xy + \frac{5}{18}y^2\end{array} \\
\underline{\frac{1}{4}x^3 + \frac{1}{6}x^2y} \\
-\frac{1}{6}x^2y + \frac{1}{36}xy^2 \\
\underline{-\frac{1}{6}x^2y - \frac{1}{9}xy^2} \\
\qquad\quad \frac{5}{36}xy^2 + \frac{1}{6}y^3 \\
\qquad\quad \underline{\frac{5}{36}xy^2 + \frac{5}{54}y^3} \\
\qquad\qquad\qquad \frac{4}{54}y^3 \text{ or } \frac{2}{27}y^3
\end{array}$$

Remr. $\frac{2}{27}y^3$. **Ans.**

§ 19. Non-terminating Quotients.

There frequently arise cases of division in algebra where the quotient is non-terminating, just as in arithmetic the quotient may be a non-terminating decimal.

Exercise 25.

Divide to four terms in the quotient—

A. 1. 1 by $a + x$.
2. x by $x^2 + 1$.
3. $1 + x$ by $1 - x + x^2$.
4. $a + b$ by $ab + 1$.

B. 1. 3 by $x + c$.
2. $x - 1$ by $x^2 + 1$.
3. $x^2 + 1$ by $x^3 + 1$.
4. $1 + 2y$ by $1 - 3y$.

C. 1. $a + x$ by $a^2 + x^2$.

2. $x + 1$ by $x^2 - 1$.

3. y^3 by $1 - y^4$.

4. $1 + x^2$ by $1 - x$.

Model Solution.

Divide to four terms in the quotient ax^2 by $a^2x^3 - a^3x + a^4$.

$$\begin{array}{ll}
ax^2 & \left(\dfrac{a^2x^3 - a^3x + a^4}{\dfrac{1}{ax} + \dfrac{1}{x^3} - \dfrac{a}{x^4} - \dfrac{2a^2}{x^6} + \text{\&c. } \textbf{Ans.}} \right. \\
ax^2 - a^2 + \dfrac{a^3}{x} & \\
\hline
a^2 - \dfrac{a^3}{x} & \\
a^2 - \dfrac{a^3}{x^4} + \dfrac{a^4}{x^3} & \\
\hline
-\dfrac{a^3}{x} + \dfrac{a^3}{x^4} - \dfrac{a^4}{x^3} & \\
-\dfrac{a^3}{x} + \dfrac{a^4}{x^3} - \dfrac{a^5}{x^4} & \\
\hline
-\dfrac{2a^4}{x^3} + \dfrac{a^3}{x^4} + \dfrac{a^5}{x^4} & \\
-\dfrac{2a^4}{x^3} + \dfrac{2a^5}{x^5} - \dfrac{2a^6}{x^6} & \\
\hline
\end{array}$$

Chap. VIII.—MULTIPLICATION AT SIGHT.

§ **20.** It is of the very greatest importance for the student to be able to write down at once, without going through all the steps, the product of certain forms of algebraic expressions. These we shall now consider.

§ **21.** The product of two binomials of the form

$(x + a)$ and $(x + b)$ is $x^2 + (a + b)\,x + ab$.

Here we see that—

(1) the product of two binomial factors consists of three terms;

(2) the first term of the product is the square of the first term of each factor;

(3) the co-efficient of the second term of the product is the *sum* of the second terms of the factors;

(4) the last term is the product of the second terms of the factors.

If **a** or **b** or both have the − sign, then by ordinary multiplication we get.

$$(\mathbf{x}+\mathbf{a})\,(\mathbf{x}-\mathbf{b})=\mathbf{x}^{2}+(\mathbf{a}-\mathbf{b})\,\mathbf{x}-\mathbf{ab}$$

and $$(\mathbf{x}-\mathbf{a})\,(\mathbf{x}-\mathbf{b})=\mathbf{x}^{2}-(\mathbf{a}+\mathbf{b})\,\mathbf{x}+\mathbf{ab}.$$

Here we see that the same four laws hold good for the formation of the product; attention being paid to the *signs* of **a** and **b**.

Examples: $(x+2y)\,(x+3y)=x^2+5yx+6y^2$:
$(x+5y)\,(x-2y)=x^2+3yx-10y^2$.

§ **22**. Consider the product of three binomials of the form

$$(\mathbf{x}+\mathbf{a}),\ (\mathbf{x}+\mathbf{b}),\ \text{and}\ (\mathbf{x}+\mathbf{c}).$$

By ordinary multiplication and the use of brackets, we find it to be—

$$\mathbf{x}^{3}+(\mathbf{a}+\mathbf{b}+\mathbf{c})\,\mathbf{x}^{2}+(\mathbf{ab}+\mathbf{ac}+\mathbf{bc})\,\mathbf{x}+\mathbf{abc}.$$

Here we see that—

(1) the number of terms in the product is one more than the number of factors;

(2) the first term is the cube of the first term of the factors;

(3) the co-efficient of the second term is the sum of the second terms of all the factors;

(4) the co-efficient of the third term is the sum of the products of the second terms of the factors taken two and two every possible way;

(5) the last term is the product of all the second terms of the factors.

NOTE.—If the sign of **a, b**, or **c**, be —, we have only to change the sign or signs accordingly in the product; thus

the product of $(\mathbf{x} + \mathbf{a})(\mathbf{x} - \mathbf{b})(\mathbf{x} - \mathbf{c})$ is

$$\mathbf{x}^3 + (\mathbf{a} - \mathbf{b} - \mathbf{c})\mathbf{x}^2 + (-\mathbf{ab} - \mathbf{ac} + \mathbf{bc})\mathbf{x} + \mathbf{abc}.$$

It is clear that the laws of formation of the product are the same for two factors as for three factors; and that we must pay attention to the signs.

Examples: 1. $(x + 2a)(x + 3a)(x + 4a) = x^3 + (2a + 3a + 4a)x^2 + (6a^2 + 8a^2 + 12a^2)x + 24a^3$
$= x^3 + 9ax^2 + 26a^2x + 24a^3.$

2. $(x + 7a)(x - 2a)(x - 3a) = x^3 + (7a - 2a - 3a)x^2 + (-14a^2 - 21a^2 + 6a^2)x + 42a^3$
$= x^3 + 2ax^2 - 29a^2x + 42a^3.$

With a little practice the student may do the additions and subtractions of the first step *mentally* and write down the product at once. It is most valuable that facility should be early acquired with these operations.

[The student will learn afterwards that these laws hold good for any number of binomial factors: but at present he need not go beyond three factors.]

§ **23**. Consider the product of $(\mathbf{a} + \mathbf{b})$ and $(\mathbf{a} - \mathbf{b})$. It is $\mathbf{a}^2 - \mathbf{b}^2$. *Hence the product of the sum and difference of two quantities is the difference of their squares.*

This is a particular case of the formula in § 21 given above; and it is of the greatest use in algebra.

Examples: 1. $(3a-4b)(3a+4b)=9a^2-16b^2$.

2. $(1+x)(1-x)=1-x^2$.

NOTE.—A useful result derived from this is—

$$(a^2+ab+b^2)(a^2-ab+b^2)=(a^2+b^2)^2-a^2b^2=a^4+a^2b^2+b^4$$

Example: $(x^2+x+1)(x^2-x+1)=x^4+x^2+1$.

§ **24.** Another particular case and no less important is

$$(a+b)^2=a^2+b^2+2ab.$$

$$(a-b)^2=a^2+b^2-2ab.$$

Hence:—*The square of the sum of two quantities is equal to the sum of the squares of the quantities, together with twice their product.*

§ **25.** Again by simple multiplication, or by the methods given above, we have—

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$$

$$(a-b-c)^2=a^2+b^2+c^2-2ab-2ac+2bc.$$

The square of the sum of several quantities is equal to the sum of the squares of each of the quantities, together with twice their product taken two and two every possible way.

Examples—

1. $(2x+3y+4z)^2=4x^2+9y^2+16z^2+12xy+16xz+24yz$.
2. $(2x+3y-4z)^2=4x^2+9y^2+16z^2+12xy-16xz-24yz$.

§ **26.** Lastly, we have another important case:—

$$(a+b)^3=a^3+b^3+3ab(a+b).$$

$$(a-b)^3=a^3-b^3-3ab(a-b).$$

The cube of the sum of two quantities is equal to the sum of the cubes of the quantities, together with three times their sum into their product.

Examples—

1. $(2x + 3y)^3 = 8x^3 + 27y^3 + 18xy\,(2x + 3y)$
$= 8x^3 + 27y^3 + 36x^2y + 54xy^2.$

2. $(4m - 5n)^3 = 64m^3 - 125n^3 - 60mn\,(4m - 5n)$
$= 64m^3 - 125n^3 - 240m^2n + 300mn^2.$

§ **27**. It is to be carefully noted that all the above results are quite **general,** that is true for all values of **a, b, c**; and they are often useful also in arithmetic.

Examples—

1. $998^2 = (1000 - 2)^2 = 1000000 + 4 - 4000 = 996004.$
2. $998 \times 1002 = (1000 - 2)\,(1000 + 2) = 1000000 - 4 = 999996.$
3. $(a+b+c)^3 = (a+b)^3 + c^3 + 3(a+b)c(a+b+c)$
$= a^3 + b^3 + 3ab(a+b) + c^3 + 3c(a+b)^2 + 3c^2(a+b)$
$= a^3 + b^3 + 3a^2b + 3ab^2 + c^3 + 3a^2c + 3b^2c + 6abc + 3ac^2 + 3bc^2.$
or $= a^3 + b^3 + c^3 + 3a^2b + 3ab^2 + 3a^2c + 3ac^2 + 3b^2c + 3bc^2 + 6abc.$
or $= a^3 + b^3 + c^3 + 3ab(a+b) + 3ac(a+c) + 3bc(b+c) + 6abc.$

EXAMPLES ON CHAP. VIII.

Multiplication at Sight.

Exercise 26.

Write down the product of—

A. **1.** $(x + 11)\,(x + 9)$; also of $(x - 6)\,(x + 4)$.
2. $(x + 3a)\,(x + 2b)\,(x + 2c)$.
3. $(y + 2m)\,(y - 3m)\,(y - 4m)$.
4. $(x - 7)\,(x - 8)\,(x - 9)$.

B. **1.** $(x + 6)\,(x + 8)$; also of $(x + 3)\,(x - 10)$.
2. $(x + 4m)\,(x + 2m)\,(x + 6m)$.
3. $(y + 3p)\,(y - 7p)\,(y - 4p)$.
4. $(s - 2a)\,(s - 2b)\,(s - 2c)$.

C. 1. $(y+4)(y+8)$; also of $(y+6)(y-11)$.

2. $(p+2q)(p+3r)(p+4s)$.

3. $(x-c)(x-7c)(x-4c)$.

4. $(a-5)(a+6)(a-7)$.

Model Solution.

Write down the product of $(3x+4y)(3x-6y)(3x+5y)$.

By Formula, the product

$$= (3x)^3 + (4y - 6y + 5y)(3x)^2 + \{4y \times (-6y) + 4y \times 5y + (-6y) \times 5y\} \times 3x + (4y) \times (-6y)(5y)$$

$$= 27^3 + 3y \times 9x^2 + (-24y^2 + 20y^2 - 30y^2) \times 3x - 120y^3$$

$$= 27x^3 + 27x^2y - 102xy^2 - 120y^3. \quad \textbf{Ans.}$$

Exercise 27.

Write down the product of—

A. 1. $(3x+5y)^2$; also of $(4x-7y)^2$.

2. $(5y-9)(5y+9)$; also of $(2x-3z)(2x+3z)$.

3. $(1+2x+3x^2)^2$; also of $(1-2x-3x^2)^2$.

4. $(x^2+xy+y^2)(x^2-xy+y^2)$.

B. 1. $(4a+3b)^2$; also of $(3m-5n)^2$.

2. $(6a-10b)(6a+10b)$; also of $(3y-4c)(3y+4c)$.

3. $(1+y+2y^2)^2$; also of $(1-y-2y^2)^2$.

4. $(a^2-ab+b^2)(a^2+ab+b^2)$.

C. 1. $(5y+3x)^2$; also of $(8a-4b)^2$.

2. $(3x-4y)(3x+4y)$; also of $(5p-2q)(5p+2q)$.

3. $(a^2+3a+2)^2$; also of $(3x^2-3x-1)^2$.

4. $(a^2+a+1)(a^2-a+1)$.

Model Solution.

Write down the product of

$$(a^2 - \tfrac{1}{3}a + \tfrac{1}{2}) \times (a^2 + \tfrac{1}{3}a - \tfrac{1}{2}).$$

By Formula this is $= a^4 - (\frac{1}{3}a - \frac{1}{2})^2 = a^4 - \frac{1}{9}a^2 + \frac{1}{3}a - \frac{1}{4}$. **Ans.**

Exercise 28.

Write down the product of—

A. 1. $(2x + 3y)^3$; also of $(3x - 2y)^3$.

2. $(3m + 4n)(9m^2 - 12mn + 16n^2)$; also of $(1 + m)(m^2 - m + 1)$.

3. $(a^2 - ab)(a^4 + a^3b + a^2b^2)$; also of $(p - q)(p^2 + pq + q^2)$.

4. $(x^2 + y^2)(x^4 - x^2y^2 + y^4)$; also of $(x^3 - y^3)(x^6 + x^3y^3 + y^6)$.

B. 1. $(4a + 2b)^3$; also of $(2x - 3c)^3$.

2. $(5x + 3y)(25x^2 - 15xy + 9y^2)$; also of $(a - 1)(a^2 + a + 1)$.

3. $(m^2 - p)(m^4 + m^2p + p^2)$; also of $(3c - 2d)(9c^2 + 6cd + 4d^2)$.

4. $(a^2 + 4b^2)(a^4 - 4a^2b^2 + 16b^4)$; also of $(1 - p^3)(1 + p^3 + p^6)$.

C. 1. $(5a + 3x)^3$; also of $(3y - 4)^3$.

2. $(2p + 5q)(4p^2 - 10pq + 25q^2)$; also of $(3p - 1)(9p^2 + 3p + 1)$.

3. $(4x^2 - 3)(16x^4 + 12x^2 + 9)$; also of $(3y^2 + 2)(9y^4 - 6y^2 + 4)$.

4. $(2 + y^2)(4 - 2y^2 + y^4)$; also of $(m^2 - 1)(m^4 + m^2 + 1)$.

Model Solution.

Write down the product of—

$$(a^2 - 3ab)(a^2 + 3ab)(a^8 + 9a^6b^2 + 81a^4b^4)$$

This is $= (a^4 - 9a^2b^2)(a^8 + 9a^6b^2 + 81a^4b^4)$

which is an example of the form $(\mathbf{a} - \mathbf{b})(\mathbf{a}^2 + \mathbf{ab} + \mathbf{b}^2)$

where $\mathbf{a} = a^4$, and $\mathbf{b} = 9a^2b^2$.

$$\text{Hence product} = \mathbf{a}^3 - \mathbf{b}^3 = (a^4)^3 - (9a^2b^2)^3$$
$$= a^{12} - 729a^6b^6.\quad \textbf{Ans.}$$

Chap. IX.—DIVISION AT SIGHT.

§ 28. It is obvious that the results we have given above for *Multiplication at Sight* may also be used for *Division at Sight.*

Product ÷ Multiplier = Multiplicand.
Product ÷ Multiplicand = Multiplier.

Hence we have—

(i) $(x^2 + (a + b)x + ab) \div (x + a) = x + b.$

(ii) $(x^2 + (a - b)x - ab) \div (x - b) = x + a.$

(iii) $(x^2 - (a + b)x + ab) \div (x - a) = x - b.$

(iv) $(x^2 - (a + b)x + ab) \div (x - b) = x - a.$

(v) $(x^2 - y^2) \div (x - y) = x + y.$
$(x^2 - y^2) \div (x + y) = x - y.$

(vi) $(x^2 + 2xy + y^2) \div (x + y) = x + y.$

(vii) $(x^2 - 2xy + y^2) \div (x - y) = x - y.$

These results are of constant use in algebra.

§ 29. Important results in Division of Binomials.

I. By ordinary division we find—

$$(x^3 - y^3) \div (x - y) = x^2 + xy + y^2.$$
$$(x^4 - y^4) \div (x - y) = x^3 + x^2y + xy^2 + y^3.$$

Now if any number of such divisions be tried, the student will find that the following laws always hold :—

(i) If the divisor is of the form $\mathbf{x - y}$ while the dividend is of the form $\mathbf{x^n - y^n}$, where $\mathbf{n}$ is any integer odd or even, the quotient comes without any remainder.

(ii) All terms of the quotient are positive.

(iii) The quotient is a homogeneous quantity of $\mathbf{n - 1}$ dimensions, the power of $\mathbf{x}$ decreasing by unity and that of $\mathbf{y}$ increasing by unity in each new term.

II. Similarly, it will be found by ordinary division, that

$$(x^4 - y^4) \div (x + y) = x^3 - x^2y + xy^2 - y^3.$$
$$(x^5 - y^5) \div (x + y) = x^4 - x^3y + x^2y^2 - xy^3 + y^4 - \frac{2y^5}{x + y}.$$

And any number of such divisions will always be found to obey the following laws :—

(i) An expression of the form $\mathbf{x^n - y^n}$ is exactly divisible by $\mathbf{x + y}$ only if $\mathbf{n}$ is an even integer.

(ii) The terms of the quotient are alternately + and —.

(iii) The quotient is a homogeneous one of $\mathbf{n - 1}$ dimensions, the powers of $\mathbf{x}$ and $\mathbf{y}$ following the same law as in the last case.

III. In like manner we find by division that—

$$(x^3 + y^3) \div (x + y) = x^2 - xy + y^2.$$
$$(x^4 + y^4) \div (x + y) = x^3 - x^2y + xy^2 - y^3 + \frac{2y^4}{x + y}.$$

Here we see that

(i) Any expression of the form $\mathbf{x^n + y^n}$ is exactly divisible by $\mathbf{x + y}$ only if $\mathbf{n}$ is an *odd integer ;*

(ii) The terms of the quotient, as in last case, are + and — alternately.

IV. Lastly we find by division that

$$(x^3 + y^3) \div (x - y) = x^2 + xy + y^2 + \frac{2y^3}{x - y}.$$

$$(x^4 + y^4) \div (x - y) = x^3 + x^2y + xy^2 + y^3 + \frac{2y^4}{x - y}.$$

Here we see that

(i) An expression of the form $\mathbf{x^n + y^n}$ is never exactly divisible by $\mathbf{x - y}$, but always leaves a remainder of the form $\frac{2y^n}{x - y}$;

(ii) The terms of the quotient are all +.

These results in division are of such importance that we shall give them summed up briefly, as follows :—

I. $\mathbf{x^n - y^n}$ is **always** divisible exactly by $\mathbf{x - y}$, if $\mathbf{n}$ is *any integer ;* and all terms of the quotient are +.

II. $\mathbf{x^n - y^n}$ is exactly divisible by $\mathbf{x + y}$ only if $\mathbf{n}$ is an *even integer ;* and the terms of the quotient are alternately + and —.

III. $\mathbf{x^n + y^n}$ is exactly divisible by $\mathbf{x + y}$ only if $\mathbf{n}$ is an *odd integer ;* and the terms of the quotient are alternately + and —.

IV. $\mathbf{x^n + y^n}$ is **never** exactly divisible by $\mathbf{x - y}$ [but always leaves a remainder $+ \frac{2y^n}{x + y}$].

EXAMPLES ON CHAP. IX.

Division at Sight.

Exercise 29.

Write down the quotient of—

A. 1. $9a^2 - 4b^2$ by $3a - 2b$; $16x^4 - 9y^2$ by $4x^2 + 3y$.

2. $p^4 - 16$ by $p^3 - 2p^2 + 4p - 8$; and $\frac{1}{8}x^3 - \frac{1}{27}$ by $\frac{1}{2}x - \frac{1}{3}$.

3. $x^{10} + y^{10}$ by $x^2 + y^2$; and $\dfrac{1}{y^4} - \dfrac{1}{z^4}$ by $\dfrac{1}{y} - \dfrac{1}{z}$.

4. $81m^4 - 256n^4$ by $3m - 4n$.

B. 1. $81x^2 - 36y^4$ by $9x + 6y^2$; and $25m^4 - 49n^6$ by $5m^2 - 7n^3$.

2. $a^8 - 81$ by $a^2 - 3$; and $\dfrac{1}{p^3} + \dfrac{1}{q^3}$ by $\dfrac{1}{p} + \dfrac{1}{q}$.

3. $m^{12} - p^8$ by $m^3 - p^2$; and $\dfrac{1}{a^6} + \dfrac{1}{b^9}$ by $\dfrac{1}{a^2} + \dfrac{1}{b^3}$.

4. $16a^4x^4 - 81b^4y^4$ by $2ax - 3by$.

C. 1. $4m^2 - 9n^4$ by $2m + 3n^2$; and $36a^6 - 25b^4$ by $6a^3 - 5b^2$.

2. $x^4 - 16y^4$ by $x - 2y$; and $p^4 - 16q^4$ by $p^3 - 2p^2q + 4pq^2 - 8q^3$.

3. $m^{16} - 81n^{16}$ by $m^4 - 3n^4$; and $a^{25} + b^{25}$ by $a^5 + b^5$.

4. $p^{12} - 1$ by $p^3 - 1$; and $x^9 + y^{12}$ by $x^3 + y^4$.

Model Solution.

Write down the quotient of $a^6 - 64b^6$ by $a^2 + 2ab + 4b^2$.

By Formula $a^6 - 64b^6 = (a^3 - 8b^3)\ (a^3 + 8b^3)$

$= (a - 2b) \times (a^2 + 2ab + 4b^2) \times (a^3 + 8b^3)$

Hence $(a^6 - 64b^6) \div (a^2 + 2ab + 4b^2) = (a - 2b) \times (a^3 + 8b^3)$

$= a^4 - 2a^3b + 8ab^3 - 16b^4$. **Ans.**

Exercise 30.

Write down by inspection the quotient of—

A. 1. $16x^4 + 36x^2y^2 + 81y^4$ by $4x^2 + 6xy + 9y^2$.

2. $a^3 + b^3 - 3ab + 1$ by $a + b + 1$.

3. $m^8 + m^4 + 1$ by $m^4 - m^2 + 1$.

4. $m^8 + m^4 + 1$ by $m^4 - m^2 + 1$.

B. 1. $a^4 + 4a^2b^2 + 16b^4$ by $a^2 - 2ab + 4b^2$.

2. $x^3 + 8y^3 + 1 - 6xy$ by $x + 2y + 1$.

3. $16a^4 + 4a^2 + 1$ by $4a^2 + 2a + 1$.

4. $x^3 + m^3 + a^3b^3 - 3abmx$ by $x + m + ab$.

C. 1. $256p^4 + 144p^2q^2 + 81q^4$ by $16p^2 - 12pq + 9q^2$.

2. $x^3 + y^3 - 6xy + 8$ by $x + y + 2$.

3. $81a^4 + 9a^2 + 1$ by $9a^2 - 3a + 1$.

4. $x^3 - y^3 + 27z^3 + 9xyz$ by $x - y + 3z$.

Model Solution.

Write down the quotient of—

$8p^6 + 27q^3 + 64m^3 - 72p^2qm$ by $2p^2 + 3q + 4m$.

This is an example of the Formula

$$\frac{\mathbf{x}^3 + \mathbf{y}^3 + \mathbf{z}^3 - 3\mathbf{xyz}}{\mathbf{x} + \mathbf{y} + \mathbf{z}} = \mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2 - \mathbf{xy} - \mathbf{xz} - \mathbf{yz}$$

where $\mathbf{x} = 2p^2$, $\mathbf{y} = 3q^2$, and $\mathbf{z} = 4m$.

Hence quotient $= (2p^2)^2 + (3q)^2 + (4m)^2 - (2p^2)(3q) - (2p^2)(4m) - (3q)(4m)$

$4p^4 + 9q^2 + 16m^2 - 6p^2q - 8p^2m - 12qm$. **Ans.**

Chap. X.—FACTORISATION.

§ **30.** It is of the utmost importance in algebra that the student should learn to resolve a complex expression into its simple component factors and to be able to detect readily whether it is resolvable or not. For this purpose the rules of **Multiplication and Division at Sight**, as given above, are of the greatest use.

§ **31**. If we see by inspection that each of the terms of an expression contains a common factor, we get at once a factor of the whole expression, which we may place outside a bracket thus :—

$$\mathbf{ax + bx + cx = x\,(a + b + c).}$$

Exercise 31.

Resolve into factors—

A. **1.** $3x^2 - 15x$.
2. $4m^4 - 16m^2 - 8m$.
3. $9a^5 - 3a^3 + 24a^2$.
4. $5a^6x^3 - 15a^3x^4 + 25a^4x^5$.

B. **1.** $4a^3 - 12a^5$.
2. $9x^2 - 18x^4 + 27x^6$.
3. $10p^8 + 15p^4 - 20p^5$.
4 $6a^6x^2 + 12a^3x^4 - 18a^4x^6$.

C. **1.** $6b^5 - 18b^7$.
2. $3a^5 - 12a^4 + 21a^6$.
3. $7a^8b^2 - 21a^4b^4 - 35a^6b^6$.
4. $5m^3n^4 - 10m^4n^5 + 15m^5n^6$.

Model Solution.

Resolve into factors—

$$16p^6q^3 - 48p^3q^5 + 64p^4q^6.$$

Since 16, & p^3, & q^3, divide each term, it is evident that the expression $= 16p^3q^3\,(p^3 - 3q^2 + 4pq^3)$. **Ans.**

§ **32.** If an expression consist of four terms which can be evidently arranged in two pairs, each pair containing some common factor, we know that the expression must be the product of two binomials, thus :—

$$\mathbf{ax + by + ay + bx = a\,(x + y) + b\,(x + y)}$$
$$\mathbf{= (a + b)\,(x + y).}$$

or we may of course say $\mathbf{= x(a + b) + y\,(a + b)}$
$$\mathbf{= (x + y)\,(a + b).}$$

So $\mathbf{ax - by + bx - ay = a(x - y) + b(x - y)}$
$$\mathbf{= (a + b)\,(x - y).}$$

The following are Examples, but practice alone will give the student proficiency :—

1. Resolve $6ax + 20by + 8ay + 15bx$. We must take together terms which have some common factor : we cannot take them as they stand ; re-arranging we get $6ax + 8ay + 20by + 15bx$ which is clearly $= 2a\,(3x + 4y) + 5b\,(4y + 3x) = (2a + 5b)\,(3x + 4y)$.
2. Resolve $15a^2 - 28by + 20ab - 21ay$. Here it is not clear which two terms are to be taken together as there are three containing the letter *a* ; but as the term $-21ay$ has also the factor *y* which is found in the term $-28by$, it is likely that these two go together: we therefore try the arrangement $15a^2 + 20ab - 28by - 21ay$. This gives $5a(3a + 4b) - 7y(4b + 3a)$, which shows a common factor is $3a + 4b$; hence given expression $= (3a + 4b)\,(5a - 7y)$.

Exercise 32.

Resolve into factors—

A. **1.** $ac + 2bd + ad + 2bc$.
2. $x^2 - mx + 4x - 4m$.
3. $y^4 + 2 + y^3 + 2y$.
4. $p^2x^2 - bq^2 - bq^2 + q^2x^2$.

B. **1.** $2ax + by + bx + 2ay$.
2. $x^2 - 5a + ax - 5x$.
3. $3y^3 + 5y^2 + 5 + 3y$.
4. $2bx^2 + 3ay^2 - 3bxy - 2axy$.

C. 1. $3am + cn + cm + 3an$.
2. $4a + b^2 + ab + 4b$.
3 $2y^4 - 2 - y^3 + 4y$.
4. $bxy - acz - bz + acxy$.

§ 33. Expression of the form $x^2 + px + q$.

If the third term **q** is the product of two factors **a** and **b** and the co-efficient of the second term **p** is the sum of **a** and **b**; then we know by the rules of multiplication given above that the expression is the product of two factors **(x + a), (x + b)**, such that **p = a + b**, and **q = ab**.

The two factors, numerical or literal, which multiplied give q and added give p must be found by trial.

The following are examples :—

1. Resolve $x^2 + 12x + 32$. We find by trial that 4 and 8 multiplied give 32, and added give 12; hence we know that $x^2 + 12x + 32 = (x + 4)(x + 8)$.
2. Resolve $x^2 + 9x + 18$. There are only two numbers 3 and 6 which multiplied give + 18 and added give 9; hence $x^2 + 9x + 18 = (x + 3)(x + 6.)$
3. Resolve $x^2 + ax(b + c) + a^2bc$. Here **q** $= a^2bc$, and **p** $= ab + ac$. Now clearly **q** $= ab \times ac$. $\therefore$ given expression $= (x + ab)(x + ac)$.

§ 34. Expression of the form $x^2 - px + q$.

Here if **q = a × b**, and **p = a + b**, then the factors are **(x − a) (x − b)**. For example—

1. Resolve $x^2 - 10x + 9$. Here **q** $= 9$, and **p** $= -10 = -(1 + 9)$; and the factors are $(x-1)(x-9)$.
2. Resolve $x^2 - (a + b + c)x + ac + bc$.
   Here **q** $= ac + bc = c(a + b)$ and **p** $= (a + b) + c$:
   $\therefore$ given expression $= (x - c)(x - \overline{a + b})$.

Exercise 33.

Resolve into factors—

A. 1. $m^2 + 8m + 15$.
2. $x^2 + 10ax + 24a^2$.
3. $y^4 - 11y^2 + 30$.
4. $a^4 + 5a^2b + 6b^2$.

B. 1. $p^2 + 8p + 12$.
2. $n^2 - 15n + 50$.
3. $x^4 + 11ax^2 + 28a^2$.
4. $c^2 + 12c + 32$.

C. 1. $x^2 + 14x + 48$.
2. $m^4 - 5m^2 + 4$.
3. $a^2 - 13ab + 30b^2$.
4. $a^2b^2 - 7abc + 10c^2$.

§ 35. Expression of the form $x^2 + px - q$ or $x^2 - px - q$.

It follows from simple multiplication that

$$x^2 - px - q = (x - a)(x + b),$$

if $q = ab$, and $-p = -a + b$, and $a > b$.

But $x^2 + px - q = (x + a)(x - b)$, if $a > b$.

If the third term is negative and is the product of two factors whose *difference* is the co-efficient of the second term, then the second terms of the binomial factors must be opposite in sign, and *the sign of the* **middle term** *of the expression will be the sign of the* **greater** *of the second terms of the factors.*

Examples.—

1. Resolve $x^2 + 4x - 12$. Here we must find two factors of -12, of opposite sign, whose sum $= +4$. Try -4 and $+3$; their sum is -1; that will not do; try $+4$ and -3; their sum is $+1$; that will not do. Again try 2 and -6; their sum is -4; that will not do; try -2 and $+6$; their sum is $+4$. That must be what we want.

$\therefore x^2 + 4x - 12 = (x - 2)(x + 6)$.

2. Resolve $x^2 - (b - c)\,x - (a - b)\,(a - c)$.

Here $\mathbf{q} = (a-b)(a-c)$, and $\mathbf{p} = b - c$: now $(a-b) - (a-c) = c - b = -p$.

Hence given expression $= (x + a - b)\,\{x - (a - c)\}$.

Exercise 34.

Resolve into factors—

A. **1.** $x^4 + 15x^2 + 56$.
2. $x^2 - 7x - 44$.
3. $x^2 + ax - 42a^2$.
4. $p^2 + 2pm - 15m^2$.

B. **1.** $a^2 - 15a + 54$.
2. $y^4 + 7y^2 - 30$.
3. $y^2 - 2aby - 8a^2b^2$.
4. $m^2 + mnx - 6n^2x^2$.

C. **1.** $x^2 - 11x + 24$.
2. $a^2y^2 + 3ay - 28$.
3. $p^2 + pqx - 6q^2x^2$.
4. $m^2x^2 + 3mxy - 54y^2$.

§ 36. Expression of the form $px^2 + qxy + ry^2$.

If the co-efficient of x is not unity, the resolution becomes much more difficult. The reason will be seen if we consider the product of the two factors $(ax + by)\,(cx + dy)$, and also of the two factors $(ax + dy)\,(cx + by)$.

The first product is $acx^2 + bdy^2 + (ad + bc)\,xy$.

The second product is $acx^2 + bdy^2 + (ab + dc)\,xy$.

The first and third terms are the same in both cases, and are the products of the factors ax, cx and by, dy respectively.

But it depends on the co-efficient of xy whether ax and by are to be taken together or ax and dy are to be taken together for one factor. Now when the co-efficients of x^2, of y^2, and of xy are *numbers*, it is not clear from inspection how we shall group the terms for the required factors: we must simply find by trial what combination satisfies the co-efficient of xy.

Hence, if the given expression is $\mathbf{px^2 + qxy + ry^2}$, where $\mathbf{p} = a \times b$, and $\mathbf{r} = c \times d$, we cannot say at once if the factors are $(ax + cy)(bx + dy)$ or $(ax + dy)(bx + cy)$. We must simply find by trial if $\mathbf{q} = ad + bc$ or $= ac + bd$.

For example take the expression $36x^2 + 77xy + 40y^2$.

Here the factors of $36x^2$ may be $4x$ and $9x$, and the factors of $40y^2$ may be $8y$ and $5y$: we must find by trial how these are to be associated

$$\left.\begin{matrix} 4x + 8y \\ 9x + 5y \end{matrix}\right\} \text{ or } \left\{\begin{matrix} 4x + 5y. \\ 9x + 8y. \end{matrix}\right.$$

By trial we see that $4 \times 8 + 9 \times 5 = 77$. Hence the required factors are $(4x + 5y)(9x + 8y)$.

But we might have taken $6x$ and $6x$, or $2x$ and $18x$ or x and $36x$ as the factors of $36x^2$; and we might have taken $4y$ and $10y$, or $2y$ and $20y$, or y and $40y$, as the factors of $40y^2$. If we please to try any of these combinations, we shall find that none of them will give 77 for the co-efficient of xy.

§ **37. Expression of the form $\mathbf{px^2 - qxy + ry^2}$.**

If $\mathbf{p} = a \times b$, and $\mathbf{r} = c \times d$; then the factors must be $(ax - cy)(bx - dy)$ or else $(ax - dy)(bx - cy)$: we must find *by trial* whether $\mathbf{q} = (ad + bc)$, or $= (ac + bd)$.

Take for example $6x^2 - 41xy + 35b^2$.

The factors may be

$$\left.\begin{matrix} 3x - 5y \\ 2x - 7y \end{matrix}\right\} \text{ or } \left\{\begin{matrix} 3x - 7y. \\ 2x - 5y. \end{matrix}\right.$$

Both give the first and last terms, but neither gives the co-efficient of xy.

The factors may be

$$\left.\begin{matrix} 2x - 35y \\ 3x - y \end{matrix}\right\} \text{ or } \left\{\begin{matrix} 2x - y. \\ 3x - 35y. \end{matrix}\right.$$

But this would give $-107xy$ or $-73xy$ for middle term, neither of which is correct.

Again we may try for the factors

$$\left.\begin{matrix} 2x - 35y \\ 6x - y \end{matrix}\right\} \text{ or } \left\{\begin{matrix} x - y. \\ 6x - 35y. \end{matrix}\right.$$

The former gives $-211xy$, and the latter $-41xy$ for the middle term. Hence the required factors are $(x-y)(6x-35y)$.

§ 38. Expression of the form $\mathbf{p}x^2 \pm \mathbf{q}xy - \mathbf{r}y^2$.

Suppose as before $\mathbf{p} = a \times b$, and $\mathbf{r} = c \times d$.

By the rules of multiplication the factors have got opposite signs for their second terms. Hence they are of the form

$$(ax - cy)(bx + dy), \text{ or } (ax + cy)(bx - dy),$$
$$\text{or } (ax - dy)(bx + cy), \text{ or } (ax + dy)(bx - cy).$$

It is best to write these down in the following way :—

$$\left.\begin{matrix} ax & cy \\ bx & dy \end{matrix}\right\} \text{ or } \left\{\begin{matrix} bx & cy \\ ax & dy \end{matrix}\right.$$

and simply find by trial whether the difference of the products ad and bc, or the difference of the products bd and ac give $\mathbf{q}$ the co-efficient of xy, the + sign or — sign being given to the larger product according as the sign of $\mathbf{q}$ is + or — .

Examples.—1. Resolve $6x^2 + 19xy - 36y^2$.

The factors may be $\left.\begin{matrix} x+6y \\ 6x-6y \end{matrix}\right\}$ but this gives $+30xy$ for middle term, which is wrong.

or, The factors may be $\left.\begin{matrix} 3x-6y \\ 2x+6y \end{matrix}\right\}$ but this gives $+6xy$ for middle term, which is wrong.

or, The factors may be $\left.\begin{matrix} 3x+9y \\ 2x-4y \end{matrix}\right\}$ or $\left\{\begin{matrix} 3x-4y. \\ 2x+9y. \end{matrix}\right.$

By trial we find the latter gives $+19xy$ as middle term, which solves the problem.

Hence the factors are :—$(3x - 4y)(2x + 9y)$.

2. Resolve $6x^2 - 19xy - 36y^2$. In the same way we may find the factors to be $(3x + 4y)(2x - 9y)$, the — sign being so placed that the larger of the cross products is — .

The solution of such problems becomes a matter of considerable difficulty if the co-efficients of x^2 and y^2 are large numbers, which admit of resolution into factors in many different ways.

The same principles as those just given will however suffice in most cases to solve the problem, and by inspection we can readily see which factors are obviously unsuitable.

Exercise 35.

Resolve into factors—

A. 1. $2a^2 + 3a + 1$.
2. $3m^2 + 10m + 3$.
3. $3x^2 + 5x - 2$.
4. $4x^2 - 22x - 42$.

B. 1. $3a^2 + 5a + 2$.
2. $2m^2 + 9m + 4$.
3. $2x^2 - 5x - 12$.
4. $5x^2 - 34x - 48$.

C. 1. $2a^2 + 5a + 2$.
2. $3m^2 + 8m + 4$.
3. $4x^2 - 19x - 5$.
4. $3x^2 - 8x - 35$.

Exercise 36.

Resolve into factors—

A. 1. $12x^2 + 34x + 24$.
2. $15a^2 - 47a + 36$.
3. $20x^2 + 9xy - 18y^2$.
4. $14a^2 - 45ab - 14b^2$.

B 1. $6a^2 + 19a + 10$.
2. $12m^2 - 34m + 24$.
3. $20a^2 + 64ab - 21b^2$.
4. $6x^2 + 5xy - 25y^2$.

C. 1. $12y^2 - 27y + 28$.
2. $12x^2 + 28x + 15$.
3. $35x^2 + 13ax - 12a^2$.
4. $33y^2 + 76yz - 32z^2$.

§ **39.** The difference of two squares can of course be at once resolved into the product of the sum and difference of two quantities; for

$$a^2 - b^2 = (a - b)(a + b).$$

Examples :—$9x^2 - 16y^2 = (3x - 4y)(3x + 4y)$;
$81x^2 - 1 = (9x - 1)(9x + 1)$.

Note 1.—One or both of the squares may be compound quantities, and the formation for the whole expression may not be quite so obvious then.

For example.—Resolve $a^2 + b^2 - c^2 - d^2 + 2ab + 2cd$. Re-arranged this becomes $a^2 + 2ab + b^2 - (c^2 - 2cd + d^2)$ *i.e.*, $(a + b)^2 - (c - d)^2$. Hence the factors will be $(a + b + c - d)(a + b - c + d)$.

Note 2.—Since $xy = \left(\frac{x + y}{2}\right)^2 - \left(\frac{x - y}{2}\right)^2$, it is clear that the product of two factors may always be thrown into the shape of the difference of two squares.

In other words, *if a given expression can be thrown into two factors, it is always possible to throw the given expression into the form of the difference of two squares.* Sometimes adding and subtracting the same number or quantity enables us to see the two squares when otherwise they are not obvious.

Examples.—1. Resolve $x^2 - 2x - 8$; by adding and subtracting 1, which cannot alter the given expression, it becomes $x^2 - 2x + 1 - 9$, that is $(x - 1)^2 - 3^2$. Hence the factors are $(x - 1 + 3)(x - 1 - 3)$, that is $(x + 2)(x - 4)$.

2. Resolve $x^4 - 10x^2 + 9$. This is obviously

$$= x^4 - 6x^2 + 9 - 4x^2 = (x^2 - 3)^2 - (2x)^2$$
$$= (x^2 - 3 - 2x)(x^2 - 3 + 2x)$$
$$= (x - 3)(x + 1)(x - 1)(x + 3).$$

Exercise 37.

Resolve into factors—

A. 1. $9x^2 - 25y^2$.
2. $121a^2 - 9b^2$.
3. $a^2 + b^2 - x^2 - 2xy + 2ab - y^2$.
4. $4a^2 - 2yz - z^2 - y^2$.

B. 1. $36p^2 - 49q^2$.
2. $9a^4 - 16b^4$.
3. $m^2 + 4n^2 - x^2 + 4mn$.
4. $4a^2 - 16c^2 + 8cd - 12ab + 9b^2 - d^2$.

C. 1. $4a^2 - 49b^4$.
2. $256y^4 - 25x^6$.
3. $1 + x^2 - 4a^2 - 9b^2 - 12ab + 2x$.
4. $4a^2 - y^2 - z^2 + 2yz$.

Exercise 38.

Resolve into factors and simplify—

A. 1. $(a+b)^2 - a^2$.
2. $(24b+c)^2 - (23b-c)^2$.
3. $(3a+7b)^2 - (2a-3b)^2$.
4. $(6x+5y)^2 - (5x-4y)^2$.

B. 1. $a^2 - (b-a)^2$.
2. $(5y+2z)^2 - (3y-z)^2$.
3. $(4x+3y)^2 - (3x-2y)^2$.
4. $(10a-4b+3)^2 - (9a-3b+3)^2$.

C. 1. $(b + 3c)^2 - 4c^2$.
2. $9a^2 - (3a - 5b)^2$.
3. $(7x + 6y)^2 - (4x - 3y)^2$.
4. $(2a - 3b + 1)^2 - (3a - 2b + 1)^2$.

§ **40.** All the formulæ given under Multiplication and Division by Inspection (Chapters VI, VII) are constantly of use for resolution into factors, thus—

(1) $\mathbf{x^3 + y^3 = (x + y)(x^2 - xy + y^2)}$.

(2) $\mathbf{x^3 - y^3 = (x - y)(x^2 + xy + y^2)}$.

(3) $\mathbf{x^4 + y^4 + x^2y^2 = (x^2 + xy + y^2)(x^2 - xy + y^2)}$.

(4) $\mathbf{x^4 + x^2 + 1 = (x^2 - x + 1)(x^2 + x + 1)}$.

(5) $\mathbf{x^6 - y^6 = (x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)}$.

It is to be noted that these formulæ are perfectly general; that is $\mathbf{x}$ and $\mathbf{y}$ may be anything. The following are examples of each of these five formulæ, which show how many cases they may apply to.

(1) $8a^3+27b^3=(2a+3b)(4a^2-6ab+9b^2)$. Here $\mathbf{x}=2a, \mathbf{y}=3b$.

(2) $125x^3-1=(5x-1)(25x^2+5x+1)$. Here $\mathbf{x}=5x, \mathbf{y}=1$.

(3) $16a^4+81b^4+36a^2b^2=(4a^2+6ab+9b^2)(4a^2-6ab+9b^2)$. Here $\mathbf{x}=2a, \mathbf{y}=3b$.

(4) $625m^4+25m^2+1=(25m^2-5m+1)(25m^2+5m+1)$. Here $\mathbf{x}=5m$.

(5) $64c^6-1=(2c+1)(2c-1)(4c^2+2c+1)(4c^2-2c+1)$. Here $\mathbf{x}=2c, \mathbf{y}=1$.

§ 41. Other Useful Formulæ are—

1. $\mathbf{x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-xz-yz)}$.

2. $\mathbf{x^3+y^3+1-3xy=(x+y+1)x^2+y^2+1-xy-x-y)}$.

Examples of these are—

1. $8a^3+27c^3+b^3-18abc=(2a+b+3c)(4a^2+b^2+9c^2-2ab-6ac-3bc)$.

This comes under the first formula by putting $2a=\mathbf{x}$, $b=\mathbf{y}$, and $3c=\mathbf{z}$.

2. $125m^3+8n^3+1-30mn=(5m+2n+1)(25m^2+4n^2+1-10mn-5m-2n)$.

This is reducible to the second formula by putting $5m=\mathbf{x}$, $2n=\mathbf{y}$.

Exercise 39.

Resolve into factors—

A. 1. $8c^3-27a^3$.
2. x^3-216y^3.
3. $16x^4+36x^2y^2+81y^4$.
4. $625m^4+25m^2+1$.

B. 1. $343a^3+512$.
2. $1000x^3-1$.
3. $81x^4+225x^2y^2+625y^4$.
4. $a^8b^8+a^4b^4+1$.

C. 1. $a^3 + 64b^3$.

2. $729x^3y^3 - 8a^3$.

3. $256x^4 + 16y^4 + 64x^2y^2$.

4. $m^{16}y^{16} + m^8y^8 + 1$.

Exercise 40.

Resolve into two or more factors—

A. 1. $500a^2b - 20b^2$.

2. $5y^4 - 15y^3 - 90y^2$.

3. $24x^4 - 4x^3 - 4x^2$.

4. $8m^4 - 32m^3 - 24m^2 + 96m$.

B. 1. $x^5 - 8x^2y^3$.

2. $y^2 + z^2 - x^2 - 2yz$.

3. $45x^2y^2 - 30xy^2 - 60x^2y + 40xy$.

4. $6p^4 - 12p^3 - 15p^2 + 30p$.

C. 1. $b^2y^3 - 16c^2y^3$.

2. $14a^2b^3 - 35a^3b^2 + 14a^4b$.

3. $20a^4b + 18a^3b^2 - 18a^2b^3$.

4. $4m^4 - 8m^3 + 24m^2 - 48m$.

Exercise 41.

Resolve into two or more factors—

A. 1. $x^3 - (a - b + c)x^2 - (ab - ac + bc)x + abc$.

2. $x^3 - 4x^2 - 11x + 30$.

3. $a^4b - a^2b^3 - a^3b^2 + ab^4$.

4. $(x^2 + y^2 - z^2)^2 - 4x^2y^2$.

B. 1. $y^3 + (a - b - c)y^2 - (ab - ac + bc)y - abc$.

2. $x^3 - 2x^2 - 31x - 28$.

3. $24a^2b^2 - 30ab^3 - 36b^4$.

4. $a^4 - b^4 + 2b(a^3 + b^3) - (a + b)^2(a - b)^2$.

C. 1. $z^3 + (a + b - c)z^2 + (ab - ac - bc)z - acb$.

2. $y^3 - 8y^2 + 4y + 48$.

3. $5x^4y^4 - 5xy$.

4. $(1 + x)^2 (1 + z^2) - (1 + z)^2 (1 + x^2)$.

Chap. XI.—THE GREATEST COMMON MEASURE.

§ **42. The Greatest Common Measure (G.C.M.)** of two or more algebraical quantities is **their Greatest Common Divisor, or Factor of Highest Degree**; and the terms **Highest Common Factor (H.C.F.)** or **Greatest Common Divisor (G.C.D.)** are often used instead of the older term **G.C.M.**

§ **43. Algebraic G.C.M. not necessarily arithmetical G.C.M.**

Two algebraic expressions may have no common factor algebraically, yet their arithmetical equivalents for any special values of the letter or letters involved may have a common factor. For example $x - 2$ and $x + 2$ have no common factor algebraically, yet if we put $x = 4$, they become 2 and 6 which have the common factor 2; and if we put $x = 10$, they become 8 and 12 respectively, which have the common factor 4. So, again, $x - y$ and $x + y$ have no common divisor or algebraic measure, but for particular numerical values of x and y, they may have an arithmetical common measure. Thus, put $x = 9$, $y = 7$, and they become 2 and 16 with 2 for G.C.M.; put $x = 10$, $y = 6$, and they become 4 and 16, with 4 for G.C.M.; put $x = 21$, $y = 9$, and they become 12 and 30, with 6 for G.C.M., and so on.

It follows clearly from this that the arithmetical values of the algebraic G.C.M. of any two expressions may not be the numerical G.C.M. of the arithmetical values of the two given expressions. Thus the greatest common algebraic factor of $3x^3 - 4x^2 - 3x - 2$ (A) and $2x^3 + 3x^2 - 5x - 18$ (B) is found to be $x - 2$. But if we put $x = 4$, the numerical equivalents of (A) and (B) are 114 and 138, whose G.C.M. is 6; whereas if we put $x = 4$, the algebraical G.C.M. $x - 2$ becomes $= 2$.

In the same way one expression may be greater algebraically, that is of *higher degree*, than another, but it is not necessarily greater arithmetically for special numerical values of the letters involved.

§ 44. To find the G.C.M. by Inspection.

This can always be done in the case of **monomials**; for we have only to look what numbers and letters are common to the two or more given expressions, *e.g.*, $27a^3b^4c^2$, $36a^5b^2c^4$, and $45ab^3c^3$: the G.C.M. is obviously $9ab^2c^2$.

Again for **polynomials** this method should always be adopted *if the given expressions can be resolved into factors by inspection*, *e.g.*, the G.C.M. of $3x^3 - 3y^3$, $12x^4 - 12y^4$, and $27x^5 - 27y^5$, is $3(x - y)$.

If the student is expert at factorisation, it will save much time and trouble to use the method of inspection as much as possible. The student should therefore remember that—

The greatest common measure, factor, or divisor (**G.C.M.**, **G.C.F.**, or **G.C.D.**) of two or more expressions is to be determined *by inspection*, in every case when the expressions can be resolved into factors by inspection.

Exercise 42.

Find, by inspection, the **G.C.M.** of—

A. **1.** $34a^2x^3y^4$ and $51a^3x^2y^3$.

2. $15a^4b^3c^2$ and $12a^2bc^2$.

3. $21a^3bx^3$, $28ab^3cx^2$ and $35a^2b^2cx^4$.

4. $21x^5y^3z$, $27x^2y^4z^3$ and $24x^4yz^5$.

B. 1. $39m^3n^5x^4$ and $52m^2n^3x^5$.
2. $49x^2yz^3$ and $63x^3y^2z^4$.
3. $18p^3q^4s^3$, $27p^2q^2s^5$ and $36p^4qs^6$.
4. $12x^5y^3z^4$, $20x^6yz^5$ and $16x^2y^4z^6$.

C. 1. $64xy^4z^7$ and $80x^2y^3z^5$.
2. $35b^2c^3d$ and $42b^3c^2d^2$.
3. $15a^3b^4c$, $25ab^3c^4$ and $35a^4b^2c^2$.
4. $20a^4b^5x^2$, $15a^5bx^9$ and $30a^3b^4x^6$.

Exercise 43.

Find, by inspection, the **G.C.M.** of—

A. 1. $x^2 + xy$ and $x^2 - y^2$.
2. $a^3 + a^2b$ and $a^3 + b^3$.
3. $x^2yz + xy^2z$ and $x^2y - y^3$.
4. $6x^2 + x - 2$ and $12x^2 - x - 6$.

B. 1. $(a + b)^2$ and $a^2 - b^2$.
2. $x^3y - xy^3$ and $x^5y^2 - x^2y^5$.
3. $m^3 + 8n^3$ and $m^2 + mn - 2n^2$.
4. $6y^2 - 3y - 9$ and $8y^2 - 18$.

C. 1. $x^2 - 4y^2$ and $x^2 + 2xy$.
2. $6a^2 - 9ab$ and $4a^2 - 9b^2$.
3. $y^2 + 3y + 2$ and $y^2 - 4$.
4. $12a^2 + 13a - 4$ and $12a^2 - 19a + 4$.

Exercise 44.

Find, by inspection, or resolution into factors, the **G.C.M.** of—

A. 1. $2a^2 + 5a - 3$, $2a^2 + 7a + 3$ and $3a^2 + 11a + 6$.
2. $2y^2 + 9y + 4$, $2y^2 + 11y + 5$ and $2y^2 - 3y - 2$.
3. $a^2b(b^2 - c^2)$, $ab(b^2 + 3bc + 2c^2)$ and $a^2(b^2 - bc - 2c^2)$.
4. $2x^2 - 11x + 5$, $3x^2 - 16x + 5$ and $4x^2 - 19x - 5$.

B. 1. $2x^2 - x - 6$, $2x^2 + 5x + 3$ and $4x^2 \pm 4x - 3$.
2. $3a^4 + 8a^3 + 4a^2$, $3a^5 + 11a^4 + 6a^3$ and $3a^4 - 16a^3 - 12a^2$.
3. $3a^2 - 7ab + 2b^2$, $3a^2 - 4ab - 4b^2$ and $2a^2 - ab - 6b^2$.
4. $6y^3(y^4 - 2y^2 - 3)$, $6y^2(y^4 - 5y^2 + 6)$, $10y(y^4 - 4y^2 + 3)$.

C. 1. $3y^2 + 10y - 8$, $12y^2 + 7y - 10$ and $6y^2 + 2y - 4$.
2. $ay^2 + 2a^2y + a^3$, $2ay^2 - 4a^2y - 6a^3$ and $3(ay + a^2)^2$
3. $x^2 - xy - 6y^2$, $x^2 + xy - 12y^2$ and $x^2 + 3xy - 18y^2$.
4. $x^3 - 10x^2 + 27x - 18$, $x^3 - 12x^2 + 41x - 30$ and $x^3 - 14x^2 + 55x - 42$.

Model Solution.

Find, by inspection, or resolution into factors, the **G.C M.** of

$$6x^3y\ (x^2y - xy^2 + x^2 - y),\quad 6x^2y^2\ (x^2 - y - x^2y + y^2)$$
$$\text{and } 6x^3y^2\ (x^3 - xy - x^2y^2 + y^3).$$

Evidently $6x^2y$ divides each factor outside the brackets, and is therefore one factor of the required **G C.M.**

Also $x^3y - xy^2 + x^2 - y = xy\,(x^2 - y) + x^2 - y = (xy + 1)(x^2 - y)$
and $x^2 - y - x^2y + y^2 = (x^2 - y) - y\,(x^2 - y) = (1 - y)(x^2 - y)$.
and $x^3 - xy - x^2y^2 + y^3 = x\,(x^2 - y) - y^2\,(x^2 - y) = (x - y^2)(x^2 - y)$
From this we see that $x^2 - y$ is the only other common factor.
Hence the **G.C.M.** required is $6x^2y\,(x^2 - y)$. **Ans.**

§ 45. G.C.M. of two complex Polynomials.

If the given algebraical expressions are not simple enough to be at once factorised, then a method similar to the arithmetical method of finding the G.C.M. is employed. The method may be summarised as follows :—

Divide the greater quantity (*i.e.*, the one with highest index of the principal letter) by the less : then divide the former divisor by the remainder ; and go on in this way till the division comes without an remainder. Then the last divisor in the process is the **G.C.M.** of the two given expressions.

Illustrative Examples are given in Exercises 45—47.

The **reason for this process** depends on the simple fact that whatever divides each of two quantities A and B, must also divide the sum or the difference of those quantities: and must also divide the sum or difference of any multiples of those quantities. For if x is a common divisor of A and B, and if $A = mx$, and $B = nx$; then it is obvious that

$$A \pm B = mx \pm nx = x(m \pm n).$$

Also $pA \pm qB = pmx \pm qnx = x(pm \pm qn)$.

§ 46. The Introduction or Exclusion of Factors.

There is often a great saving of labour if we multiply or divide the remainder at any stage of the operation by some factor which will make the division come exactly for the highest terms and so bring the operation more rapidly to an end. The only condition to be attended to is, that **the factor introduced or excluded in this way must not be any factor of the G. C. M. required.** For example in the *Model Solution* of Exercise 45 we have excluded the factor 5 from the remainder at one stage of the process, and the factor 47 at another; but neither of these is any factor of the required G. C. M. for it is not common to all terms of the given expressions. Again, in the *Model Solution* to Exercise 46, we exclude the numerical factor 48; and, in the *Model Solution* to Exercise 47, we introduce the factor 3 on two different occasions and exclude the factor 10 on another.

Exercise 45.

Find, by the ordinary method, the **G. C. M.** of—

A. 1. $6a^2 + 51a + 99$ and $3a^2 + 57a + 144$.

2. $36y^2 + 96y - 36$ and $28y^2 + 100y + 48$.

3. $a^5 - 5a^3 + 5a^2 - 1$ and $a^4 + a^3 - 4a^2 + a + 1$.

4. $18x^3 - 18x^2y + 6xy^2 - 6y^3$ and $60x^2 - 75xy + 15y^2$.

B. 1. $8y^2 + 58y + 77$ and $6y^2 + 55y + 176$.
2. $63a^2 + 144a - 45$ and $54a^2 - 63a + 15$.
3. $y^4 + 4y^2 + 16$ and $y^5 + y^4 - 2y^3 + 17y^2 - 10y + 20$.
4. $6y^4 + 29x^2y^2 + 9x^4$ and $3y^3 - 15xy^2 + x^2y - 5x^3$.

C. 1. $12b^2 - 15b - 63$ and $8b^2 + 74b + 105$.
2. $3x^2 + 15x - 72$ and $4x^2 + 92x + 480$.
3. $a^6 - 6a^4 + 9a^2 - 4$ and $a^6 + a^5 - 2a^4 + 3a^2 - a - 2$.
4. $2p^3 + 10p^2 + 14p + 6$ and $p^3 + p^2 + 7p + 39$.

Model Solution.

Find, by the ordinary rule, the **G. C. M.** of $6x^3 + 7x^2 - 29x + 12$ and $12x^3 - 22x^2 + 23x - 20$.

$$
\begin{array}{r|c|c|l}
 & 6x^3 + 7x^2 - 29x + 12 & 12x^3 - 22x^2 + 23x - 20 & 2 \\
 & 6 & 12x^3 + 14x^2 - 58x + 24 & \\
\cline{2-3}
-x & 36x^3 + 42x^2 - 174x + 72 & -36x^2 + 81x - 44 & \\
 & 36x^3 - 81x^2 + 44x & & \\
\cline{2-2}
-3 & 123x^2 - 218x + 72 & & \\
 & 108x^2 - 243x + 132 & & \\
\cline{2-2}
 & 5)15x^2 + 25x - 60 & -36x^2 - 60x + 144 & -12 \\
\cline{3-3}
 & 3x^2 + 5x - 12 & 47)141x - 188 & \\
 & & 3x - 4 & \\
x & 3x^2 - 4x & & \\
\cline{2-2}
3 & 9x - 12 & & \\
 & 9x - 12 & & \\
\end{array}
$$

Hence, the required **G. C. M.** is $3x - 4$. **Ans.**

Exercise 46.

Find, by the ordinary method, the **G. C. M.** of—

A. 1. $a^3 + a^2 - 10a + 8$ and $a^3 + 2a^2 - 13a + 10$.
2. $b^2 + 4b^2 - 5b - 20$ and $b^3 + 6b^2 - 5b - 30$.
3. $2y^2 + 4y^2 - 7y - 14$ and $6y^3 - 10y^2 + 21y + 35$.
4. $x^3 + 14ax^2 + 50a^2x + 7a^3$ and $x^2 + 6ax - 7a^2$.

B. 1. $y^3 - 6y^2 - 86y + 35$ and $y^3 - 5y^2 - 99y + 40$.
2. $3a^4 - 3a^2 - 2a^2 - a - 1$ and $6a^4 - 3a^3 - a^2 - a - 1$.

3. $y^5 - y^4 + y^3 + 2y^2 - 6y - 4$ and $y^5 - y^4 + y^3 - 2y^2 - 2y$.
4. $2x^4 - 4x^3 + 8x^2 - 12x + 6$ and $3x^4 - 3x^3 - 6x^2 + 9x - 3$.

C. 1. $s^3 + 3s^2 - 8s - 24$ and $s^3 + 2s^2 - 8s - 16$.
2. $2b^4 - 2b^3 + b^2 + 3b - 6$ and $4b^4 - 2b^3 + 3b - 9$.
3. $6x^3 + 13x^2y - 9xy^2 - 10y^3$ and $9x^3 + 12x^2y - 11xy^2 - 10y^3$.
4. $6y^3 + 16y^2 - 12y + 2$ and $15y^3 - 5y^2 + 12y - 4$.

Model Solution.

Find, by the usual rule, the **G. C. M.** of $2x^3 - 7x^2 - 46x - 21$ and $2x^4 + 11x^3 - 13x^2 - 99x - 45$.

$$
\begin{array}{r|l|r|l}
 & 2x^3 - 7x^2 - 46x - 21 & 2x^4 + 11x^3 - 13x^2 - 99x - 45 & x \\
 & & 2x^4 - 7x^3 - 46x^2 - 21x & \\
\cline{3-3}
 & & 18x^3 + 33x^2 - 78x - 45 & 9 \\
 & & 18x^3 - 63x^2 - 414x - 189 & \\
\cline{3-3}
x & 2x^3 + 7x^2 + 3x & 48\,)\,96x^2 + 336x + 144 & \\
\cline{2-2}
-7 & -14x^2 - 49x - 21 & 2x^2 + 7x + 3 & \\
 & -14x^2 - 49x - 21 & & \\
\end{array}
$$

Hence, required **G. C. M.** is $2x^2 + 7x + 3$. **Ans.**

---

§ 47. G. C. M. of three or more Expressions.

The algebraic method is just like the arithmetical one. We first find the G. C. M. of any two; then the G. C. M. of this G. C. M. so found and the third expression; and so on for any number. The last G. C. M. so found is of course the G. C. M. of *all* the given expressions.

Exercise 47.

Find, by the ordinary method, the **G. C. M.** of—

A. 1. $a^3 + 7a^2 - a - 7$, $a^3 + 5a^2 - a - 5$ and $a^2 - 2a + 1$.
2. $y^3 - 4y^2 + 9y - 10$, $y^3 + 2y^2 - 3y + 20$ and $y^3 + 5y^2 - 9y + 35$.

3. $12x^2 + 7xy - 10y^2$, $15x^2 + 2xy - 8y^2$
and $15x^2 + 5xy - 10y^2$.

4. $6x^4 + 3x^3 - 14x^2 - 4x + 8$, $3x^4 + 6x^3 + 2x^2 - 8x - 8$
and $9x^4 - 3x^3 - 6x^2 - 4x - 8$.

B. 1. $x^3 - x^2 - x + 1$, $3x^2 - 2x - 1$ and $x^3 - x^2 + x - 1$.

2. $a^3 - 7a^2 + 16a - 12$, $3a^3 - 14a^2 + 16a$ and
$5a^3 - 10a^2 + 7a - 14$.

3. $y^2 + y - 30$, $y^2 + 11y + 30$ and $y^2 - y - 42$.

4. $6x^3 - 5x^2 - 8x + 3$, $8x^3 - 6x^2 - 11x + 3$
and $4x^3 - 11x + 3$.

C. 1. $a^3 + 4a^2 - 5$, $a^3 - 3a + 2$ and $a^3 + 4a^2 - 8a + 3$.

2. $x^2 + xy + y^2$, $x^3 + 2x^2y + 2xy^2 + y^3$
and $x^4 + x^2y^2 + y^4$.

3. $x^3 + x^2y + xy^2 + y^3$, $x^3 + 3x^2y + 3xy^2 + y^3$
and $x^3 + x^2y - xy^2 - y^3$.

4. $x^4 - 9x^3 + 29x^2 - 39x + 18$, $4x^3 - 27x^2 + 58x - 39$
and $x^3 - 8x^2 + 19x - 12$.

Model Solution.

Find the **G. C. M.** of $3a^3 - a^2 - 4a + 10$, $6a^3 + 7a^2 + a + 10$
and $3a^3 + 11a^2 + 4a - 10$.

First find the **G. C. M.** of the first two expressions.

$$
\begin{array}{r|l|r|l}
 & 3a^3 - a^2 - 4a + 10 & 6a^3 + 7a^2 + a + 10 & 2 \\
 & 3 & 6a^3 - 2a^2 - 8a + 20 & \\
\cline{2-3}
a & 9a^3 - 3a^2 - 12a + 30 & 9a^2 + 9a - 10 & \\
 & 9a^3 + 9a^2 - 10a & & \\
\cline{2-2}
 & -12a^2 - 2a + 30 & & \\
 & 3 & & \\
\cline{2-2}
-4 & -36a^2 - 6a + 90 & & \\
 & -36a^2 - 36a + 40 & 9a^2 + 15a & 3a \\
\cline{2-3}
 & 10\,)\,30a + 50 & -6a - 10 & -2 \\
\cline{2-2}
 & 3a + 5 & -6a - 10 & \\
\end{array}
$$

We find it to be $3a + 5$.

Taking $3a + 5$ and $3a^3 + 11a^2 + 4a - 10$, we find

$$\begin{array}{r|l}
3a^3 + 11a^2 + 4a - 10 & 3a + 5 \\ \cline{2-2}
3a^3 + 5a^2 \phantom{{}+ 4a - 10} & a^2 + 2a - 2 \\ \cline{1-1}
6a^2 + 4a \phantom{{}- 10} & \\
6a^2 + 10a \phantom{{}- 10} & \\ \cline{1-1}
-6a - 10 & \\
-6a - 10 & \\ \cline{1-1}
\end{array}$$

or $3a + 5$ is a divisor of the 3rd expression.

Hence the **G.C.M.** of the three given expressions is $3a + 5$. **Ans.**

Chap. XII.—LEAST COMMON MULTIPLE.

§ **48. Definition of L. C. M.** The Algebraical Least Common Multiple (L.C.M.) of two or more quantities is the smallest expression that contains each of the given quantities exactly. The same remarks apply here that we made under G.C.M. as to the arithmetical value of the L.C.M. not necessarily being the L.C.M. of the arithmetical values of the given expressions for any particular values of the letters involved.

§ **49. To find the L. C. M. by Inspection.** Whenever it is possible, we should find the L.C.M. by Inspection—and always for Monomials.

For the Least Common Multiple of [*or smallest quantity that can be divided by each of*] two or more quantities is obviously that which contains all the factors that occur in all the quantities but with none of them repeated unnecessarily. Hence if the given quantities can be easily resolved by inspection, we write down all their component factors, and then select each factor that occurs, *taking care to repeat it as often only as it is repeated in any one of the expressions.* The product of these factors is the **L.C.M**

Thus let P, Q, R be the three given expressions whose L.C.M. is wanted; suppose P to contain the elementary factors A, B, C; Q the elementary factors B, C, D; and R the elementary factors C, D, E; then clearly the L.C.M. of P, Q, and R will be A × B × C × D × E.

Exercise 48.

Find, by inspection, the **L.C.M.** of—

A. 1. $4a^2bc$, $5ab^2c^3$, $6a^3bc^2$.
2. $2xy$, $3x^2y$, $4xy^3$, $3x^3y^2$.
3. $17mn^2p$, $34m^3n$, $51n^2p^3$.
4. $49abc^2$, $21a^2b^2c$, $56a^3bc^3$.

B. 1. $5x^4y$, $6xy^2$, $3x^2y^3$.
2. $3a^2$, $4b^2$, $6ab^3$.
3. $29abc$, $58a^2b$, $87abc^2$.
4. $13m^3n^4$, $39m^4n^3$, $52m^2n$.

C. 1. $3a^2b^3$, $2ab$, $4a^3b^4$.
2. $2xy^2$, $3x^2y$, $4x^3y^4$, $6x^4y^3$.
3. $30abc$, $45bcd^2$, $60a^2b^3d$.
4. $72x^2y^3z^2$, $36x^3y^2z^3$, $108xyz^4$.

Model Solution.

Find, by inspection, the **L.C.M.** of—

$$4x^2yz^2,\ 6xy^3z^4,\ 6x^3yz^2,\ 12x^4yz^4.$$

Evidently the L.C.M. of the numerical co-efficients is 12, of the x factors is x^4, of the y factors is y^3, of the z factors is z^4.

Hence the **L.C.M.** required is $12x^4y^3z^4$. **Ans.**

Exercise 49.

Find, by inspection, the **L.C.M.** of—

A. 1. $8x^2 - 16xy$ and $x^2 - 4xy + 4y^2$.
2. $6(a^2 + ab)$, $8(ab - b^2)$ and $10(a^2 - b^2)$.
3. $m^2 + m - 42$, $m^2 - 11m + 30$ and $m^2 + 2m - 35$.
4. $5a^2 + 16a + 3$, $5a^2 + 11a + 2$ and $a^2 + 5a + 6$.

B. 1. $6a^2b + 6ab^2$ and $9a^3 - 9ab^2$.
2. $x^2 + 4x + 3$, $x^2 + \underline{\overset{2}{3}x} - 3$ and $x^2 - 1$.
3. $2a^2 + 3a + 1$, $2a^2 + 5a + 2$ and $a^2 + 3a + 2$.
4. $2p^2 + 15p - 8$, $2p^2 + 3p - 2$ and $p^2 + 10p + 16$.

C. 1. $(x - y)^2$ and $x^3 - y^3$.
2. $x^3 - x^2y$, $x^3 - xy^2$ and $x + y$.
3. $3y^2 + 11y + 6$, $3y^2 + 8y + 4$ and $y^2 + 5y + 6$.
4. $3b^2 - 13b + 14$, $3b^2 - b - 14$ and $b^2 - 4$.

Model Solution.

Find, by inspection, the **L.C.M.** of—

$12x^2 + 3x - 42$, $12x^3 + 30x^2 + 12x$ and $32x^2 - 40x - 28$.

Resolving into factors we have—

(i) $12x^2 + 3x - 42 = 3\ (4x^2 + x - 14) = 3\ (4x - 7)\ (x + 2)$.
(ii) $12x^3 + 30x^2 + 12x = 6x\ (2x^2 + 5x + 2) = 6x\ (2x + 1)\ (x + 2)$.
(iii) $32x^2 - 40x - 28 = 4\ (8x^2 - 10x - 7) = 4\ (4x - 7)\ (2x + 1)$.

Hence the **L.C.M.** required must contain the factors,

3, $6x$, 4, $4x - 7$, $x + 2$, and $2x + 1$

and $\therefore = 12\,x\ (4x - 7)\ (x + 2)\ (2x + 1)$. **Ans.**

§ 50. To find the L. C. M. of two or more complex Expressions.

If the resolution of the given expressions into factors cannot be effected by mere inspection; we must then have recourse to the method of finding the G. C. M. (or greatest common factor) as given above.

This will enable us to find the other factors of the given expressions: and then, as before, we have to select each factor that occurs in each expression, taking care not to repeat it oftener than it occurs in *any one* of the given expressions.

The product of all these selected factors is the required L. C. M.

Exercise 50.

Find, by the usual method, the **L. C. M.** of—

A. 1. $2x^4-7x^3+13x^2-10x+8$ and $3x^4-8x^3+7x^2+10x-8$.

2. $6x^4+8x^3+15x^2+3x-5$ and $12x^4+13x^3-16x^2-8x+5$.

3. $2a^4+a^3-20a^2-7a+24$ and $2a^4+3a^3-13a^2-7a+15$.

4. $2x^3+11x^2+23x+20$, $2x^3+9x^2+16x+15$, and $2x^3+7x^2+9x+10$.

B. 1. $3y^4+y^3-2y^2+5y-3$ and $6y^4-7y^3+11y^2-5y+3$.

2. $4a^4+7a^3-15a^2+10a-12$ and $8a^4+2a^3+29a^2-3a+18$.

3. $6x^2+33xy+15y^2$ and $3x^2y^2+14xy^3-5y^4$.

4. $2a^4-a^3+5a^2-3a-3$, $2a^4+a^3+4a^2+3a-6$ and $3a^4-a^3+11a^2-3a+6$.

C. 1. $2x^4+5x^3-8x^2-11x+12$ and $2x^4+7x^3-5x^2-14x+8$.

2. $6a^2+38a-28$ and $27a^3+27a^2-30a$.

3. $2y^4-5y^3+y^2+23y-30$ and $6y^4-20y^3+39y^2-19y+15$.

4. $3x^3-x^2+x+2$, $3x^3-6x^2+6x-3$ and $3x^3-9x^2+9x-6$.

Model Solution.

Find, by the rule, the **L.C.M.** of $6x^3-13x^2-13x+30$ and $8x^3+38x^2+59x+30$.

$$
\begin{array}{r|c|r|l}
 & 8x^3+38x^2+59x+30 & 6x^3-13x^2-13x+30 & \\
1 & 6x^3-13x^2-13x+30 & & \\
\cline{2-2}
 & x\,)2x^3+51x^2+72x & & \\
\cline{2-2}
 & 2x^2+51x+72 & & \\
 & & 6x^3+153x^2+216x & 3x \\
\cline{3-3}
 & & -166x^2-229x+30 & -83 \\
 & & -166x^2-4233x-5976 & \\
\cline{3-3}
 & & 2002)\underline{4004x+6006} & \\
x & 2x^2+3x & 2x+3 & \\
\cline{2-2}
24 & 48x+72 & & \\
 & 48x+72 & & \\
\end{array}
$$

Hence, **G. C. M.** of the two expressions is $2x + 3$.

Also, by simple division,

1st expn. $= (2x + 3) \times (4x^2 + 13x + 10)$.
2nd expn. $= (2x + 3) \times (3x^2 - 11x + 10)$.

Hence, required **L.C.M.** $= (2x+3)(4x^2+13x+10)(3x^2-11x+10)$. **Ans.**

§ **51**. The following method is sometimes given for finding the L.C.M. of two expressions. **Divide the Product of the two given Expressions by their G.C.M.** It is easy to see that this agrees with the other method. For if **d** be the G.C.M. of two quantities **A** and **B**, suppose **A = pd**, and **B = qd**. Then **p** and **q** have of course no common factor: hence, by last method, L.C.M. of **A** and **B = pqd**.

But this = **pd × qd ÷ d, i.e., A × B ÷ d**.

The student is however recommended to use the method of selection of factors as explained above in § 50 rather than this method.

Chap. XIII.—FRACTIONS.

§ **52**. Algebraical Fractions are treated by methods identical with those employed in Arithmetic. Hence, it is unnecessary to do more than summarise them.

§ **53**. **To reduce a fraction to lowest terms.**

If the numerator and denominator can, by inspection, be resolved into their elementary factors, then by inspection we may strike out all factors common to both.

Otherwise, we find the **G.C.M.** of numerator and denominator, and divide each by that.

The following are Examples :—

Reduce to lowest terms the following :—

(1) $\dfrac{9x^5y^2z^6}{24x^4y^3z^8}$ and (2) $\dfrac{x^3 - 5x^2 + 7x - 3}{x^3 - 3x + 2}$.

(1) It is clear that for the first fraction $3x^4y^2z^6$ is the greatest common divisor of numerator and denominator; dividing by this, we get $\frac{3x}{8yz^2}$. **Ans.**

(2) As the greatest common divisor of the numerator and denominator of the second fraction is not *obvious*, we find it by the ordinary rule for **G.C.M.** thus:—

$$\begin{array}{r|r|r|l}
1 & x^2 - 5x^2 + 7x - 3 & x^3 - 3x + 2 & \\
 & x^2 \qquad\qquad - 3x + 2 & & \\
\cline{1-2}
-5 & -5x^2 + 10x - 5 & & \\
\cline{2-2}
 & x^2 - 2x + 1 & x^3 - 4x^2 + x & x \\
\cline{3-3}
 & & 2x^2 - 4x + 2 & +2 \\
 & & 2x^2 - 4x + 2 & \\
\hline
\end{array}$$

Hence it is $x^2 - 2x + 1$.

Dividing numerator and denominator by this, we have

$$x^2 - 2x + 1\left|\frac{x^3 - 5x^2 + 7x - 3}{x^3 - 3x + 2}\right| = \frac{x - 3}{x + 2}.\ \textbf{Ans.}$$

Exercise 51.

Reduce to lowest terms the following fractions:—

A. 1. $\frac{4x^3y^2z^5}{6x^4yz^5}$; $\frac{2ab^3c^4}{14a^2b^2c}$.

2. $\frac{a^3 - 2ab^2}{a^4 - 4a^2b^2 + 4b^4}$; $\frac{(mn - 3n^2)^2}{m^3n^2 - 27n^5}$.

3. $\frac{27x + x^4}{18x - 6x^2 + 2x^3}$.

4. $\frac{x^2y^2 - 16y^2}{x^2y + 9xy + 20y}$.

B. 1. $\frac{21ab^2c}{28a^2bc^2}$; $\frac{35m^3np^2}{49m^2n^2p^2}$.

2. $\frac{3a^2 + 6a}{a^2 + 4a + 4}$; $\frac{a^3b + 2a^2b + 4ab}{a^3 - 8}$.

3. $\frac{3m^2 + 23m + 14}{3m^2 + 41m + 26}$.

4. $\frac{3x^4 + 9x^3y + 6x^2y^2}{x^4 + x^3y - 2x^2y^2}$.

C. 1. $\frac{10abx^2}{15a^2bx}$; $\frac{14x^3yz^4}{21x^4y^2z^3}$.

2. $\frac{a^2 - 2a + 1}{3a^2 + 7a - 10}$; $\frac{x^3 - x^2y - xy^2 - 2y^3}{x^3 + 3x^2y + 3xy^2 + 2y^3}$.

3. $\frac{3x^3 - 3x^2y + xy^2 - y^3}{4x^2 - 5xy + y^2}$.

4. $\frac{m^3 + 2m^2 - 13m + 10}{m^3 + m^2 - 10m + 8}$.

§ 54. To reduce a mixed quantity to a fractional form.

We have only to multiply up by the denominator as in arithmetic and simplify thus: $\mathbf{a + \frac{b}{c} = \frac{ac + b}{c}}$.

For Example—

Reduce to a purely fractional form $1 + 4x^2 + \frac{6x^4}{1 - x^2}$.

Proceeding as in arithmetic, we have

$$\frac{(1 + 4x^2)(1 - x^2) + 6x^4}{1 - x^2} = \frac{1 + 3x^2 - 4x^4 + 6x^4}{1 - x^2}.$$

$$= \frac{1 + 3x^2 + 2x^4}{1 - x^2}. \quad \textbf{Ans.}$$

Exercise 52.

Reduce to purely fractional form the following mixed quantities :—

A. 1. $2a - \frac{1}{3}$; and $4x + \frac{2}{y}$.

2. $2a + \frac{3 - 2a}{5}$ and $5y + \frac{4 - 7y}{3}$.

3. $a + b - \frac{2ab - b^2}{a + b}$.

4. $x + y - \frac{x^2 - y^2}{x + 2y}$.

B. 1. $3m - \frac{2}{3}$; and $5y + \frac{2}{x}$.

2. $3x - \frac{ax + x^2}{2a}$; and $5y - \frac{2ay - y^2}{3y}$.

3. $1 - a + a^2 - \frac{a^2}{1 + a}$.

4. $x^3 - x^2y - \frac{x^4 + y^4}{x + y}$.

C. 1. $5x - \frac{3}{4}$; and $3a + \frac{1}{a}$.

2. $2y - \frac{6y + 3y^2}{5b}$; and $6a - \frac{3y + 4}{6}$.

3. $1 - 3x + 4x^2 - \frac{4x^2}{1 + x}$.

4. $a - b + c - \frac{4bc}{a + b - c}$.

§ 55. To reduce an improper fraction to a mixed quantity.

As in arithmetic we simply divide numerator by denominator thus :—

$$\frac{ax + b}{x} = a + \frac{b}{x}.$$

An algebraic fraction is improper if the power of x, or of the leading letter, in the numerator is as high as, or high than, that in the denominator: in other words, if the *dimensions* of the numerator be equal to, or greater than, those of the denominator.

For Example—

Reduce to the form of a mixed quantity $\dfrac{4a^2 - 10ax + 4x^2}{2a + x}$.

Dividing we get $4a^2 + 10ax + 4x^2 \left(\begin{array}{l} 2a + x \\ \hline 2a - 6x \end{array} \right.$

$$\begin{array}{rrr}
4a^2 + 2ax & & \\ \hline
& -12ax + & 4x^2 \\
& -12ax - & 6x^2 \\ \hline
& & +10x^2
\end{array}$$

$2a - 6x + \dfrac{10x^2}{2a + x}$. **Ans.**

Exercise 53.

Reduce the following fractions to the form of mixed quantities:—

A. 1. $\dfrac{8a^2 + 5ab}{4a^2}$.

2. $\dfrac{4a^2 - 10ax + 3x^2}{2a}$.

3. $\dfrac{7x^3 + 6x^2y}{x^2 + xy}$.

4. $\dfrac{a^4 + x^4}{a + x}$.

B. 1. $\dfrac{9m^2 + 3mn}{3m^2}$.

2. $\dfrac{4a^2 - 12ax + 3x^3}{2a - 3x}$.

3. $\dfrac{6a^2x - 5ax^2 + 4x^3}{2a - x}$.

4. $\dfrac{1 - 2y + y^2}{1 + y}$.

C. 1. $\dfrac{15x^2 + 6xy + y^2}{3x}$.

2. $\dfrac{19ax - 6a^2}{3a - x}$.

3. $\dfrac{a^2 - 3a - 3}{a - 2}$.

4. $\dfrac{x^2}{1 + x}$.

§ 56. To reduce fractions to the Least Common Denominator.

We proceed as in arithmetic, that is, we find the **L.C.M.** of all the denominators; we put this for the common denominator; we divide this in succession by each fraction-denominator and multiply the quotient by the corresponding numerator: thus,

$\frac{\mathbf{a}}{\mathbf{bc}}, \frac{\mathbf{d}}{\mathbf{ce}}, \frac{\mathbf{f}}{\mathbf{eg}}$, will become, if **b, c, e, g** are elementary factors, $\frac{\mathbf{aeg, dbg, fbc}}{\mathbf{bceg}}$.

For Example—Reduce to **L.C.D.** and add

$$\frac{3a + 5b}{2ab} + \frac{4a + 3c}{3ac} + \frac{3b + 4c}{4bc}.$$

The **L.C.M.** of the denominators is $12abc$: hence we shall have :—

Given Expression

$$= \frac{6c\,(3a + 5b) + 4b\,(4a + 3c) + 3a\,(3b + 4c)}{12abc}$$

$$= \frac{18ac + 30bc + 16ab + 12bc + 9ab + 12ac}{12abc}$$

$$= \frac{25ab + 30ac + 42bc}{12abc}. \quad \textbf{Ans.}$$

Exercise 54.

Reduce the following to common denominators and add :—

A. 1. $\frac{a - 1}{2} + \frac{a + 3}{5} + \frac{a + 7}{10}$.

2. $\frac{3x - 1}{4} + \frac{2x - 3}{8} + \frac{4x - 7}{10}$.

3. $\frac{3x + 2y}{xy} + \frac{4x - 6z}{xz} + \frac{5y - 3z}{yz}$.

4. $\frac{a + 3}{2a} + \frac{a^2 - 6}{4a^2} + \frac{9 - 4a^3}{6a^3}$.

B. 1. $\frac{y + 6}{3} + \frac{2y + 5}{6} + \frac{3y - 7}{9}$.

2. $\frac{4a - 5}{7} + \frac{5a + 9}{14} + \frac{6a - 11}{21}$.

3. $\frac{2a+3b}{ab} + \frac{3a+4c}{ac} + \frac{2b+5c}{bc}.$

4. $\frac{3m+1}{m} + \frac{4m^2-1}{m^2} + \frac{3-m^3}{m^3}.$

C. 1. $\frac{x-5}{2} + \frac{x-4}{3} + \frac{x+9}{8}.$

2. $\frac{2y-5}{4} + \frac{3y-7}{8} + \frac{5y+12}{12}.$

3. $\frac{p+q}{2pq} + \frac{q+r}{3qr} + \frac{p+r}{4pr}.$

4. $\frac{x+6}{x} + \frac{3x^2-4}{2x^2} + \frac{5-2x^3}{3x^3}.$

§ 57. To add or subtract fractions.

We proceed as in arithmetic and reduce to equivalent fractions with least common denominator, and add or subtract the numerators.

For Example:—

Add and reduce to its simplest form—

$$\frac{3}{8(a-x)} + \frac{1}{8(a+x)} + \frac{a-x}{8(a^2+x^2)}.$$

Adding the first *two* fractions first, we get

$$\frac{3a+3x+a-x}{8(a^2-x^2)} = \frac{4a+2x}{8(a^2-x^2)}.$$

Adding $\frac{a-x}{8(a^2+x^2)}$ to this, we get

$$\frac{(4a+2x)(a^2+x^2)+(a-x)(a^2-x^2)}{8(a^2-x^2)(a^2+x^2)}$$

$$=\frac{4a^3+4ax^2+2a^2x+2x^3+a^3-a^2x-ax^2+x^3}{8(a^4-x^4)}$$

$$=\frac{5a^3+3ax^2+a^2x+3x^3}{8(a^4-x^4)}. \text{ Ans.}$$

Exercise 55.

Add, and reduce to simplest form—

A. 1. $\frac{a+2}{a+3}+\frac{a+3}{a-4}$.

2. $\frac{x+a}{x-a}+\frac{x-a}{x+a}+\frac{x^2-a^2}{x^2+a^2}$.

3. $\frac{4}{x-2}+\frac{7}{3x-6}+\frac{5}{4x-8}$.

4 $\frac{12}{x^2+10x+21}+\frac{15}{x^2+9x+14}+\frac{6}{x^2+5x+6}$.

B. 1. $\frac{3}{x+5}+\frac{6}{x-3}$.

2. $\frac{x}{x+y}+\frac{y}{x-y}+\frac{x^2-y^2}{x^2+y^2}$.

3. $\frac{7}{2a-3}+\frac{3}{4a-6}+\frac{9}{8a-12}$.

4 $\frac{8}{x^2-3x+2}+\frac{10}{x^2-x-2}+\frac{4}{x^2-1}$.

C 1. $\frac{2x}{1-x^2}+\frac{2x}{1+x^2}$.

2. $\frac{a+b}{a-b}+\frac{a-b}{a+b}+\frac{a^2+b^2}{a^2-b^2}$.

3. $\frac{x}{x+2y}+\frac{3y}{2x+4y}+\frac{2x+5y}{3x+6y}$.

4. $\frac{9}{4x^2+8x+3}+\frac{11}{6x^2+7x-3}+\frac{13}{6x^2+x-1}$.

Exercise 56.

Find the value of—

A. 1. $\frac{a+3}{a+4}-\frac{a+2}{a+5}$.

2. $\frac{2x-3}{x-4}-\frac{2x-4}{x-3}$.

3. $\dfrac{x+2y}{x-2y} - \dfrac{x-2y}{x+2y} + \dfrac{8xy}{(x-2y)^2}$.

4. $\dfrac{8}{3-6x} - \dfrac{8}{3+6x} - \dfrac{4}{3-12x^2}$.

B. 1. $\dfrac{4m^2+n^2}{4m^2-n^2} - \dfrac{2m-n}{2m+n}$.

2. $\dfrac{1}{2a-3b} - \dfrac{a+b}{4a^2-9b^2}$.

3. $\dfrac{x^2-yz}{yz} - \dfrac{xz-y^2}{xz} - \dfrac{xy-z^2}{y}$.

4. $\dfrac{1}{a(a+b)} - \dfrac{1}{b(b-a)}$.

C. 1. $\dfrac{2x+5}{2x+4} - \dfrac{2x+4}{2x+5}$.

2. $\dfrac{2xy}{x^2-y^2} - \dfrac{x}{x-y} - \dfrac{y}{x+y}$.

3. $\dfrac{a}{a^3+b^3} - \dfrac{b}{a^3-b^3} - \dfrac{ab(a^2+b^2)}{a^6-b^6}$.

4. $\dfrac{1}{3(a-3)} - \dfrac{1}{6(a^2-9)} - \dfrac{a}{3(a^2-9)}$.

Exercise 57.

Find the value of :—

A. 1. $\dfrac{2x+5z}{x^2z} + \dfrac{4z-3z^2}{xz^2} - \dfrac{5xz-2z^2}{x^2z^2}$.

2. $\dfrac{3}{x^2-7x+12} + \dfrac{6}{x^2-11x+28} - \dfrac{5}{x^2-10x+21}$.

3. $\dfrac{2-5a}{a+3} - \dfrac{3+a}{3-a} + \dfrac{2a(2a-11)}{a^2-9}$.

4. $\dfrac{a}{a^2-b^2} - \dfrac{b}{a^2+b^2} + \dfrac{a^3+b^3}{b^4-a^4} + \dfrac{ab}{(a+b)(a^2+b^2)}$.

B. 1. $\dfrac{2}{x^2-1}+\dfrac{3}{2x+1}-\dfrac{2x+3}{2x^2+3x+1}$.

2. $\dfrac{6}{x^2-x-12}-\dfrac{4}{x^2+8x+15}+\dfrac{2}{x^2+x-20}$.

3. $\dfrac{y}{x(x^2-y^2)}+\dfrac{x}{y(x^2+y^2)}-\dfrac{x^4+y^4}{xy(x^4-y^4)}$

4. $\dfrac{1}{(a^2-y^2)(x^2+y^2)}+\dfrac{1}{(y^2-a^2)(x^2+a^2)}$
$+\dfrac{1}{(x^2+a^2)(x^2+y^2)}$.

C. 1. $\dfrac{3}{a+b}+\dfrac{5}{a-b}-\dfrac{6a}{a^2-b^2}$.

2. $\dfrac{x+2a}{x-a}-\dfrac{x-2a}{x+a}+\dfrac{6x^2}{x^2-a^2}$.

3. $\dfrac{1}{4x^3(x+y)}-\dfrac{1}{4x^3(y-x)}+\dfrac{1}{2x^2(x^2+y^2)}-\dfrac{x^4}{x^8-y^8}$.

4. $\dfrac{a}{(a-b)(a-c)}-\dfrac{b}{(a-b)(b-c)}+\dfrac{c}{(c-a)(c-b)}$.

Model Solution.

Find the value of

$$\frac{2a+y}{(x-a)(a-b)}+\frac{a+b+y}{(x-b)(b-a)}-\frac{x+y-a}{(x-a)(x-b)}.$$

The L. C. M. of all the denominators is $(x-a)(x-b)(a-b)$.

Reducing to this as a common denominator, we have—

$$\frac{(2a+y)(x-b)-(a+b+y)(x-a)-(x+y-a)(a-b)}{(x-a)(x-b)(a-b)}$$

$$=\frac{2ax+xy-2ab-by-ax-bx-xy+a^2+ab+ay-ax-ay+a^2+bx+by-ab}{(x-a)(x-b)(a-b)}$$

$$=\frac{2a^2-2ab}{(x-a)(x-b)(a-b)}=\frac{2a}{(x-a)(x-b)}.$$ **Ans.**

§ 58. To Multiply or Divide fractions.

Here also we proceed exactly as in Arithmetic, and resolve all numerators and denominators into elementary factors, so as to cancel like factors and reduce to simplest form.

For example—

Multiply, reducing to simplest form—

$$\frac{a^2 - 18a + 80}{a^2 - 5a - 50} \times \frac{a^2 - 6a - 7}{a^2 - 15a + 56} \times \frac{a + 5}{a - 1}.$$

Resolving into factors we get

$$\frac{(a - 8)\,(a - 10)}{(a + 5)\,(a - 10)} \times \frac{(a + 1)\,(a - 7)}{(a - 8)\,(a - 7)} \times \frac{a + 5}{a - 1}.$$

$$= \frac{a + 1}{a - 1}. \quad \textbf{Ans.}$$

Exercise 58.

Multiply, expressing the result in the simplest form :—

A. 1. $\dfrac{2y}{x - 2y} \times \dfrac{x^2 - 4y^2}{xy \times 2y^2}.$

2. $\dfrac{a^2 + 5a + 4}{a^2 + 7a + 12} \times \dfrac{a^2 + 3a + 2}{a^2 + 2a + 1}.$

3. $\dfrac{x^2 - a^2}{x^2 - 3ax + 2a^2} \times \dfrac{ax - 2a^2}{x^2 + ax} \times \dfrac{x^2 - ax}{(x - a)^2}.$

4. $\dfrac{y^4 - 8y}{y^2 - 4y - 5} \times \dfrac{y - 5}{y^2 + 2y + 4} \times \dfrac{y^2 + 2y + 1}{y^3 - y^2 - 2y}.$

B. 1. $\dfrac{2a + 1}{ab + 3} \times \dfrac{a^2b^2 + 3ab}{4a^2 - 1}.$

2. $\dfrac{y^2 + 7y + 6}{y^2 + 12y + 11} \times \dfrac{y^2 - 121}{y^2 - 36}.$

3. $\dfrac{c^2 + 10c + 24}{c^2 - 14c + 48} \times \dfrac{c^3 - 8c^2}{c^2 + 6c} \times \dfrac{c^2 - 7c + 6}{c^2 + 3c - 4}.$

4. $\dfrac{x^2 - 2x - 15}{x^2 + 3x + 2} \times \dfrac{x^2}{x + 3} \times \dfrac{x^2 + 5x + 4}{x^2 - x} \times \dfrac{x^2 + x - 2}{x^2 - x - 20}.$

C. 1. $\dfrac{y^2 - 11y + 30}{y^2 - 6y + 9} \times \dfrac{y^2 - 3y}{y^2 - 5y}$.

2. $\dfrac{a^2 - 3a - 4}{a^2 - 7a - 8} \times \dfrac{a^2 - 64}{a^2 - 16}$.

3. $\dfrac{(x + y)^2 - z^2}{x^2 - (y - z)^2} \times \dfrac{z^2 - (x - y)^2}{z^2 - (x + y)^2}$.

4. $\dfrac{p^2 - 9p + 8}{p^2 + 4p - 5} \times \dfrac{p^2 - 25}{p^2 - 1} \times \dfrac{p^2 - 8p - 9}{p^2 - 17p + 72}$.

Exercise 59.

Divide, expressing the result in the simplest form :—

A. 1. $\dfrac{a^4 - 27a}{2a^2 - 5a}$ by $\dfrac{2a^2 - 11a + 15}{4a^2 - 25}$.

2. $\dfrac{y^2 + 3y + 2}{y^2 + 9y + 20}$ by $\dfrac{y^2 + 5y + 6}{y^2 + 7y + 12}$.

3. $\left\{\dfrac{x^2 + 5xy - 4y^2}{x^2 - 16y^2} - \dfrac{2xy}{2x^2 + 8xy}\right\}$ by

$$\frac{x^4 - 7x^3y + x^2y^2}{x^3 - 11x^2y + 29xy^2 - 4y^3}.$$

4. $\left\{\dfrac{2a}{2a - 3} - \dfrac{5}{6a + 9} - \dfrac{4(3a + 2)}{3(4a^2 - 9)}\right\}$ by

$$\frac{48a^3 + 44a^2 + 8a + 35}{8a^2 - 2a - 15}.$$

B. 1. $\dfrac{4y^2 + y + 14}{16y^2 - 49}$ by $\dfrac{2y^2 + 5y + 2}{2y^2 - y - 1}$.

2. $\dfrac{a^3 - a^2b - ab^2 - 2b^3}{a^3 + 3a^2b + 3ab^2 + 2b^3}$ by $\dfrac{a^3 + 2a^2 - 13a + 10}{a^3 + a^2 - 10a + 8}$.

3. $\left\{\dfrac{y}{x(x^2-y^2)}+\dfrac{x}{y(x^2+y^2)}\right\}$ by $\left(\dfrac{1}{x(x-y)}+\dfrac{1}{y(x+y)}\right)$.

4. $\left\{\dfrac{3}{x-2}+\dfrac{2}{3x+6}+\dfrac{5x}{x^2-4}\right\}$ by $\dfrac{52x^2+67x+21.}{4x^2+11x+6}$

C. 1. $\dfrac{m^3-6m^2+36m}{m^2-49}$ by $\dfrac{m^4+216m}{m^2-m-42}$.

2. $\dfrac{5x^3+2x^2-15x-6}{7x^3-4x^2-21x+12}$ by $\dfrac{4x^4+11x^2+25}{4x^4-9x^2+30x-25}$.

3. $\left(\dfrac{x+a}{x-2a}-\dfrac{x^2+2a^2}{x^2-4a^2}\right)$ by $\left(\dfrac{x^2-4a^2}{x^2-2ax}-\dfrac{x+4a}{x+2a}\right)$.

4. $\left\{\dfrac{x}{x^3+y^3}-\dfrac{y}{x^3-y^3}+\dfrac{x^3y+xy^3}{x^6-y^6}\right\}$ by

$$\frac{x^3+x^2y+xy^2+y^3}{x^2+xy+y^2}.$$

Model Solution.

Divide and reduce to simplest form :—

$$\frac{a^2+2ab+b^2-c^2}{a^2-b^2-c^2-2bc} \text{ by } \frac{b^2-2bc+c^2-a^2}{a^2-2ac+c^2-b^2}.$$

Given Expn. $=\dfrac{(a+b)^2-c^2}{a^2-(b+c)^2}\div\dfrac{(b-c)^2-a^2}{(a-c)^2-b^2}$

$=\dfrac{(a+b+c)(a+b-c)}{(a+b+c)(a-b-c)}\times\dfrac{(a-c+b)(a-c-b)}{(b-c-a)(b-c+a)}$

$=\dfrac{a+b-c}{a-b-c}\times\dfrac{a-c-b}{b-c-a}=\dfrac{a+b-c}{b-c-a}$. **Ans.**

SIMPLIFICATION OF COMPLEX FRACTIONS.

§ 59. All simplifications of fractions are ultimately reducible to the preceding rules. The following points, however, merit the careful attention of the student :—

1. Since a fraction represents the quotient of the numerator by the denominator, it is often convenient, in simplifying complex fractional expressions, to reduce them to the form of numerator ÷ denominator, and simplify each of these separately before proceeding to the final step of division.

2. The **reciprocal of a fraction,** that is, unity divided by a fraction, is equal to the fraction inverted.

3. The signs of *both* numerator and denominator of a fraction may be changed.

4. In simplifying expressions containing several improper algebraic fractions, it is often better to reduce these all to mixed numbers as the first step in simplification.

5. **Continued fractions** are of the form $\mathbf{a} + \cfrac{1}{\mathbf{b} + \cfrac{1}{\mathbf{c} + \cfrac{1}{\mathbf{d}}}}$.

To simplify such, we begin with the lowest fraction and proceed step by step, using rule 2 above, thus :—

$$= a + \cfrac{1}{b + \cfrac{1}{\cfrac{cd+1}{d}}}$$

$$= a + \cfrac{1}{b + \cfrac{d}{cd+1}} = a + \cfrac{1}{\cfrac{bcd+b+d}{cd+1}}$$

$$= a + \frac{cd+1}{bcd+b+d} = \frac{abcd+ab+ad+cd+1}{bcd+b+d}$$

§ 60. In simplifying expressions, it is to be noted that if one factor of the expression becomes $=0$, or **vanishes**, the whole expression of which it is a factor vanishes also. We must, however, carefully distinguish between the *vanishing* of a factor and the *cancelling* of a factor by division : in the latter case the factor is merely reduced to unity : and if all the factors in the numerator and denominator of a fraction *cancel*, the whole does not *vanish* but reduces to unity.

The following are Examples of simplification :—

(1) Simplify $4\left(\frac{2a^2}{b^2}+\frac{b}{2a}\right)^2-\left(\frac{4a^2}{b^2}-\frac{b}{a}\right)^2$.

Since this is the difference of two squares, we have

$$\text{Given Expn.}=\left\{2\left(\frac{2a^2}{b^2}+\frac{b}{2a}\right)+\left(\frac{4a^2}{b^2}-\frac{b}{a}\right)\right\}\times$$
$$\left\{2\left(\frac{2a^2}{b^2}+\frac{b}{2a}\right)-\left(\frac{4a^2}{b^2}-\frac{a}{b}\right)\right\}$$
$$=\left\{\frac{4a^2}{b^2}+\frac{b}{a}+\frac{4a^2}{b^2}-\frac{b}{a}\right\}\left\{\frac{4a^2}{b^2}+\frac{b}{a}-\frac{4a^2}{b^2}+\frac{b}{a}\right\}$$
$$=\frac{8a^2}{b^2}\times\frac{2b}{a}=\frac{16a}{b}.\ \textbf{Ans.}$$

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(2) Simplify $\frac{63a^2b^2}{p+q}\div\left[\frac{14a\,(p-q)}{15\,(x+y)}\div\left\{\frac{4\,(x-y)}{5a^2b}\div\frac{16\,(x^2-y^2)}{7\,(p^2-q^2)}\right\}\right]$.

In replacing the ÷ signs by × signs there will obviously be *one* inversion of the 1st term within the square brackets, *two* of the 2nd term, and *three* of the 3rd term.

Hence we may at once write

$$\text{Given Expn.}=\frac{63a^2b^2}{p+q}\times\frac{15\,(x+y)}{14a(p-q)}\times\frac{4\,(x-y)}{5a^2b}$$
$$\times\frac{7\,(p^2-q^2)}{16\,(x^2-y^2)}$$
$$=\frac{63b\times 3}{2a\times 4}=\frac{189b}{8a}.\ \textbf{Ans.}$$

Exercise 60.

Simplify the following fractional expressions :—

A. 1. $$\frac{a+5+\dfrac{6}{a}}{1+\dfrac{6}{a}+\dfrac{8}{a^2}}$$

2. $$\frac{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}{1-\dfrac{x^2+y^2}{(x+y)^2}}.$$

3. $$\frac{\dfrac{a^3}{x^2-a^2}}{\dfrac{x^2-ax+a^2}{x-a}-\dfrac{x^2+ax+a^2}{x+a}}.$$

4. $$\frac{a^2-c^2}{a^2+cx+c^2}\div\frac{\left(1+\dfrac{c}{a}\right)^3\left(1-\dfrac{c}{a}\right)^3}{a^3-c^3}.$$

B. 1. $$\frac{\dfrac{1}{y}-\dfrac{2}{y^2}-\dfrac{3}{y^3}}{y-\dfrac{9}{y}}.$$

2. $$\frac{\left(b+\dfrac{ab}{b-a}\right)\times\left(b-\dfrac{ab}{b+a}\right)}{\dfrac{a^2+b^2}{b^2-a^2}}.$$

3. $$\frac{1}{x-\dfrac{x^2-1}{x+\dfrac{1}{x-1}}}.$$

4. $$\left\{\frac{y+\dfrac{x-y}{1+xy}}{1-\dfrac{(x-y)y}{1+xy}}-\frac{x-\dfrac{x-y}{1-xy}}{1-\dfrac{x(x-y)}{1-xy}}\right\}\div\left(\frac{x}{y}-\frac{y}{x}\right).$$

C. 1. $$\frac{\dfrac{2}{1-b^2}}{\dfrac{1}{1-b}-\dfrac{1}{1+b}}.$$

2. $$1+\frac{y}{1+y+\dfrac{2y^2}{1-y}}.$$

3. $$\frac{\dfrac{a^2}{1-a^4}+\dfrac{2a^4}{1-a^8}}{\left(\dfrac{a^2+1}{a}\right)^2}.$$

4. $$\frac{\dfrac{a}{2}\left(\dfrac{1}{a-b}-\dfrac{1}{a+b}\right)\times(a+b)}{\dfrac{a^2b+ab^2}{a^2-b^2}}.$$

Chap. XIV.—GENERAL TESTS.

Exercise 61.

A. 1. Simplify $2y-\{y-(x+y)-[y-(y-x-y)]+3x\}$.

2. Find the sum of $a+b-2(c+d)$, $b+c-3(d+a)$, and $c+d-4(a+b)$, and take this sum from the difference between $2(b+d)-3(a+c)$ and $b+2c-2(a+d)$.

3. Multiply $\frac{1}{2}a+\frac{2}{3}b$ by $a-\frac{1}{3}b$, and divide the result by $\frac{1}{4}a-\frac{1}{12}b$.

4. If $x=1$, $y=-2$, $z=3$, $k=-4$, find the value of $\dfrac{x^2y^2+y^2z+k(x-y)}{10x-(y+z)^2}$.

5. Write down the square of b^2-2b+3, and the cube of a^2-5.

6. Simplify $\dfrac{2x}{x+y}+\dfrac{2y}{x-y}-\dfrac{x^2+y^2}{x^2-y^2}$.

B. **1.** From the sum of $\frac{1}{3}a^2 + \frac{1}{4}b^2 - \frac{3}{4}c^2$, $\frac{1}{2}a^2 + \frac{1}{3}b^2 + \frac{1}{2}c^2$, and $\frac{5}{6}a^2 - \frac{1}{6}b^2 - \frac{1}{3}c^2$ take the difference between $\frac{2}{3}a^2 + \frac{1}{2}b^2 - \frac{1}{3}c^2$ and $\frac{1}{2}a^2 - \frac{1}{4}b^2 + \frac{1}{2}c^2$.

2. If $x = 1$, $y = -2$, $z = 3$, $k = -4$, find the value of $\sqrt{k^2 - 4y + a^2} - \sqrt{z^3 + y^3 + x + k}$.

3. Find (i) the quotient of $a^2 - ab + \frac{3}{16}b^2$ by $a - \frac{1}{4}b$, and (ii) the product of this result by $\frac{1}{2}a + \frac{1}{4}b$.

4. Simplify $\dfrac{a^2 - 7ab + 12b^2}{a^2 + 5ab + 6b^2} \div \dfrac{a^2 - 5ab + 4b^2}{a^2 + ab - 2b^2}$

5. What is the greatest quantity that divides both $7y^3 - 10y^2 - 7y + 10$ and $2y^3 - y^2 - 2y + 1$.

6. Find the value of $\dfrac{x^2}{a^2(x-a)^2} + \dfrac{2}{x^2 + ax + a^2} - \dfrac{4x}{x^3 - a^3} - \dfrac{x^2}{ax^3 - a^4}$.

C. **1.** If $x = -\frac{1}{9}y$, find the value of $5(x - y) - 2\{3x - (x + y)\} + 7\{(x - 2y) - (5x - 2y)\}$.

2. Add together $3a + 4b + 5c$, $\dfrac{a}{3} - \dfrac{b}{4} - \dfrac{c}{5}$, and $\dfrac{10}{3}a - 2b - 3c$: and from the sum take $a + b - \dfrac{c}{3}$

3. Multiply $\frac{3}{2}a^3 - 5a^2 + \frac{1}{4}a + 9$ by $\frac{1}{2}a^2 - a + 3$.

4. Reduce to lowest terms $\dfrac{6x^4 - 2x^3 + 7x^2 - x + 2}{6x^4 - 12x^3 + 21x^2 - 6x + 9}$.

5. Find the L.C.M. of $p^2 - 1$, $7p^2 + 5p - 2$, and $7p^2 - 5p - 2$.

6. Simplify $\left(\dfrac{a-b}{a+b} + \dfrac{a+b}{a-b}\right)\left(\dfrac{a^2+b^2}{2ab} + 1\right)\dfrac{ab}{a^2+b^2}$.

Exercise 62.

A. 1. Simplify $4[4a-5-2(3a-4)+5\{2a-3-2(3a-4)\}+5\{2a-3-(2a-\overline{7a-5})\}]$.

2. If $x=5$, $y=3$, $z=1$, find the value of
$\sqrt{5xy^2}+\sqrt[3]{9yz}-2\sqrt[4]{3x+y-2z}$.

3. Divide $x^2y^3-c(x^2+c)y+c^2x$ by $xy-c$.

4. Show that $(x-y)(x+y-z)+(y-z)(y+z-x)+(z-x)(z+x-y)=0$.

5. Find the G. C. M. of $5x^3-8x^2-3x-2$ and $5x^3+2x^2-33x+18$.

6. Simplify $\dfrac{5}{3(a-1)}-\dfrac{1}{6(a+2)}-\dfrac{3}{2a}-\dfrac{2a+3}{a(a-1)(a+2)}$.

B. 1. If $x=\frac{1}{2}$, $y=2$, find the value of

$$\frac{x^3+y^3}{x^2-xy+y^2}+\frac{x^3-y^3}{x^2+xy+y^2}.$$

2. Divide $y^3+(4ac-c^2)y-(a-2c)(a^2+3c^2)$ by $y-a+2c$.

3. If $m=2$, find the numerical value of
$m^{2m-1}+2m^{m-1}+(2m-1)^m$.

4. Write down the square of $\frac{1}{2}x+\frac{1}{3}y-\frac{1}{4}z$: and the cube of $\frac{1}{2}a+\frac{1}{3}b$.

5. Find the value of $(a+b+c)(a+b-c)(a+c-b)\times(b+c-a)$ when $c^2=a^2+b^2$.

6. Simplify
$$\frac{\dfrac{x^2+y^2}{y}-x}{\dfrac{1}{y}-\dfrac{1}{x}}\times\frac{x^2-y^2}{x^3+y^3}\times\left(\frac{x+y}{x-y}+\frac{x-y}{x+y}\right)\times\left(\frac{x}{x+y}+\frac{y}{x-y}\right).$$

C. 1. Simplify $$18\left[\frac{2a}{9}-\frac{1}{6}\left(\frac{2b}{3}+c\right)+24\left\{\frac{3a}{8}-\frac{2b-3c}{12}\right\}+30\left\{\frac{7c}{15}-\frac{4}{5}(2a-b)\right\}\right].$$

2. Find the remainder when $5a^4-7a^3+3a^2-a+8$ is divided by $a-4$.

3. Find the L. C. M. of a^3+b^3, $3a^2-2ab-b^2$, and $a^3-a^2b+ab^2$.

4. Express as the product of 3 factors $(ax+by)^2+(bx-ay)^2+(ay+bx)^2+(by-ax)^2$.

5. Simplify $\dfrac{x^3+y^3}{x^4-y^4}-\dfrac{x+y}{x^2-y^2}-\dfrac{1}{2}\left\{\dfrac{x-y}{x^2+y^2}-\dfrac{1}{x-y}\right\}$.

6. The product of 2 factors is $(3a+2b)^3-(2a+3b)^3$ and one of the factors is $a-b$: find the other.

Exercise 63.

A. 1. If $a=\frac{1}{2}$, $b=2$, find the value of $$\frac{4b}{5}(b-a)-35\left[\frac{3a-4b}{5}-\frac{1}{10}\left\{3a-\frac{5}{7}(7a-4b)\right\}\right].$$

2. Find the remainder when $x^4-3x^3y+3x^2y^2-y^4$ is divided by $x^2-xy+2y^2$.

3. Resolve into factors, (i) $10y^2+79y-8$ and
(ii) $a^2-2ab-323b^2$.

4. Find the L. C. M. of $6p^3-13p^2-13p+30$, and $8p^3+38p^2+59p+30$.

5. If $x+y=1$, show that $(x^2-y^2)^2=x^3+y^3-xy$.

6. Simplify $\dfrac{a+2b}{\frac{2}{7}a-b}-\dfrac{3a^2+63ab+70b^2}{2a^2+3ab-35b^2}$.

B. 1. If $x = \frac{25}{16}$, $y = 1$, $z = \frac{3}{4}$, show that

$$(x - \sqrt{y})(\sqrt{x} + y)\sqrt{x - y} = \frac{3z^4}{\sqrt{x - z^2}}.$$

2. Write down the factors of (i) $x^2 + y^2 + 2xy + x + y$ and (ii) $3a^3 + 6a^2 - 189a$.

3. Simplify $\dfrac{a + \dfrac{b}{2}}{2a^2 + ab + \dfrac{b^2}{2}} - \dfrac{a^2 - \dfrac{b^2}{2}}{4\left(a^3 - \dfrac{b^3}{8}\right)}$.

4. Reduce to lowest terms $\dfrac{5m^3 - 14m^2 + 16}{3m^3 - 2m^2 + 16m - 48}$.

5. Write down the square of $2x^2 - 3x - 4$.

6. From $\dfrac{a + 4}{a^2 - a - 12}$ take $\dfrac{a + 3}{a^2 + a - 12}$ and divide the result by $1 + \dfrac{2(a^3 - 12)}{a^2 + 7a + 12}$.

C. 1. If $x = 1$, $y = 2$, $z = 3$, find the value of $(x^x + y^y + z^z) \times (x + y + x)^{x + z - y}$.

2. Find the G.C.M. of $2y^2 + (6a - 10b)y - 30ab$, and $3y^2 - (9a + 5b)y + 45ab$.

3. Factorise $a^3 - b^3 - a(a^2 - b^2) + b(a - b)^2$.

4. Simplify $\dfrac{(a + 1)^3 - (a - 1)^3}{(a + 1)^4 - (a - 1)^4}$.

5. Show that $\dfrac{x + y}{xy}\left(\dfrac{1}{x} - \dfrac{1}{y}\right) - \dfrac{y + z}{yz}\left(\dfrac{1}{z} - \dfrac{1}{y}\right)$ is the difference of two squares.

6. Write down the square of $3a^2 - 2a - 5$, and the cube of $\frac{1}{2}a + 1$.

Exercise 64.

A. 1. If $z = x + 2$, and $y = x - 1$, show that

$$\frac{x^2y^2z^2 - x^2y^2 - z^2 + 1}{x^2yz - \frac{z}{y} + y\left(x^2 - \frac{1}{y^2}\right)} + 1 = x^2.$$

2. Resolve into elementary factors
(i) $x^3 + 3x^2y + xy^2 - y^3$,
(ii) $x^3 + x^2y - 3xy^2 + y^3$.

3. Write down the square of $x^3 - 3x^2 + 2x - 1$, and the cube of $\frac{a}{x} + \frac{x}{a}$.

4. Find the G.C.M., and also the L.C.M. of
$21a^2 + 8ab - 4b^2$, $49a^2 - 4b^2$, $21a^2 - 20ab + 4b^2$, and $49a^2 - 28ab + 4b^2$.

5. Reduce to the simplest form

$$\frac{(a - b)(a^2 - ab + b^2)}{a^3 + b^3} \div \frac{a^3 - b^3}{(a + b)(a^2 + ab + b^2)}.$$

6. Take the sum of $\frac{2x^2}{x + 1}$ and $\frac{x + 5}{x^2 - 1}$ from $\frac{2x^2 + 1}{x - 1}$

and multiply the result by $\left\{ \dfrac{x - \dfrac{x - 1}{x + 1}}{1 + \dfrac{x + 1}{x - 1}} \div \dfrac{x - \dfrac{x + 1}{x - 1}}{1 + \dfrac{x - 1}{x + 1}} \right\}$.

B. 1. If $a + b + c = 1$, show that $a(b + c)(b^2 + c^2 - a^2) + b(c + a)(c^2 + a^2 - b^2) + c(a + b)(a^2 + b^2 - c^2) = 2abc$.

2. Resolve into factors (i) $6x^3 + 29x^2 + 26x - 21$ and (ii) $a^3 - a^2b - ab^2 + b^3$.

3. Write down the square of $\frac{1}{2}x^2 + x + \frac{1}{2}$ and the cube of $2a + \frac{1}{2}b$.

4. Simplify $\frac{c}{a+b-c} - \frac{b}{a-b+c} + \frac{a}{c-b-a}$.

5. Find the G. C. M. of $x^5 + 4x^4 + 6x^3 + 10x^2 + 5x + 6$ and $x^6 - 2x^5 - 2x^4 - 4x^3 + x^2 + 6x$.

6. Simplify $\frac{x+y}{xy}(x^2+y^2-z^2) + \frac{y+z}{yz}(y^2+z^2-x^2) + \frac{z+x}{zx}(z^2+x^2-y^2)$.

C. 1. If $a = \frac{13}{3}$, find the value of

$$\left\{\frac{a}{2} - \left(\frac{2a-3}{3} - \frac{3a-1}{4}\right)\right\} \div \frac{a-1}{2}.$$

2. Divide $p^5 - 2p^4 - 4p^3 + 13p^2 - 11p - 7$ by $p^3 - 3p + 7$.

3. Multiply $\frac{2x^3 + 19x^2y + 35xy^2}{x^3 - x^2y - 4xy^2 - 6y^3}$ by $\frac{2x^3 + 13x^2y - 15xy^2 - 126y^3}{x^2y + 6xy^2 - 7y^3}$.

4. Write down the square of $3\left(\frac{a}{b} + \frac{b}{a}\right) - 4$: and the cube of $\left(\frac{a}{2} - \frac{b}{3}\right)$.

5. Resolve into factors

(i) $2a^3 - a^2b - 2ab^2 + b^3$

(ii) $(x+y)^3 + 3xy(1-x-y) - 1$.

6. Simplify

$$\frac{\dfrac{x^3+y^3}{x^2-y^2}}{\dfrac{x^2-xy+y^2}{x-y}} \times \left\{\frac{x+y}{y} - \frac{2x}{x+y} + \frac{x^2y-x^3}{x^2y-y^3}\right\}.$$

Exercise 65.

A. 1. If $p=3, q=5, r=7$, find the value of $p^{q-2}-p^{p-1}+\sqrt[3]{r^2+3q}$.

2. If $a=\frac{2}{7}$, $b=1\frac{1}{2}$, $c=\frac{1}{3}$, find the value of $a-[2b+5\{3c-3a-4(a+b)\}]+2a-6(b+3c)$.

3. If $z=x+2$, and $y=x-1$, find the value of
$$\frac{x^2y^2z^2-x^2y^2-z^2+1}{x^2yz-\frac{z}{y}+y\left(x^2-\frac{1}{y^2}\right)}.$$

4. Add $12\{\frac{5}{6}b-\frac{1}{3}(c-\frac{1}{4}a)\}$, $\frac{1}{3}\{(2a-3b)-(6c-a)\}$ and $16\{\frac{3}{8}a-\frac{3}{4}(b-\frac{1}{2}c)\}$ and multiply the result by $\frac{5}{8}a+\frac{2}{3}b$.

5. The product of two expressions is $\frac{y^4}{9}+\frac{y^3}{6}-\frac{3y^2}{2}+2$; if one of them is $\frac{y^2}{3}-\frac{y}{2}-1$, find the other.

6. Factorise (i) $2a^3+a^2-2a-1$, and
(ii) $6x^3-7x^2+1$.

B. 1. If $m=4$, $n=3$, $s=2$, find the value of $n^s-s^n+\sqrt[n]{m^2+3n+s}$.

2. If $a=1$, $b=2$, $x=\frac{1}{6}$, $y=\frac{2}{19}$, find the value of $a-[3a-5b-\{7a-9b-11x-(13a-15b-17x-19y)\}]$.

3. If $p=\cdot 01$, $q=\cdot 002$, $r=\cdot 012$, find the value of $p^2+q^2-r^2+2pq$.

4. Take $\frac{6}{5}(x^2-2x-1)-\frac{1}{5}(x^2-7x+4)$ from $12(x^2+2x)-21(\frac{1}{3}x^2+x-\frac{1}{7})$ and multiply the result by $2x^2-2x-1$.

5. The product of two expressions is $\frac{1}{4}x^5 - \frac{4}{3}x^4 + \frac{77}{24}x^3 - \frac{43}{12}x^2 - \frac{11}{4}x + 9$; one of them is $\frac{1}{2}x^2 - x + 3$; find the other.

6. Resolve into factors—(i) $2y^3 - y^2 - y - 3$;
(ii) $b^3 - 7b + 6$.

C. 1. Find the value of $m^{2m-1} + 2m^{m-1} + (2m-1)^m$, where $m = 2$.

2. If $a = 3$, $b = -4$, find the value of

$$21\left(\frac{2a+4b}{3} - \frac{4a-b}{7}\right) - 24\left[\frac{2a-b}{12} - \left\{\frac{5a-7b}{8} - \frac{1}{6}\left(\frac{a}{2} - \frac{5a-b}{4}\right)\right\}\right].$$

3. If $l = \cdot 02$, $m = \cdot 08$, $n = \cdot 1$, find the value of $l^3 + m^3 + 3lmn - n^3$.

4. Take $\frac{1}{6}x - y - \frac{1}{3}$ from the sum of $y + \frac{1}{2} - \frac{1}{3}x$, $\frac{1}{2}x - \frac{1}{3}y + 1$, and $x + \frac{1}{2}y - \frac{1}{3}$ and multiply the result by $x - \frac{5}{6}y - \frac{3}{2}$.

5. The product of two expressions is $12x^3 - 5x^4 - \frac{31}{6}x^2 - \frac{1}{3}x + \frac{1}{6}$ and one of them is $\frac{1}{2}x^2 - x + \frac{1}{6}$; find the other.

6. Resolve into factors—(i) $9x^2 - 9x + 2$;
(ii) $a^4 - a^3 + 8a - 8$.

Exercise 66.

A. 1. Multiply $3 - 2a + \dfrac{6a^2 - 17a + 13}{3a - 4}$ by $8 - 5a + \dfrac{4a^2 - a - 8}{1 + \frac{3}{4}a}$.

2. Multiply $\dfrac{x^2 - \frac{1}{x}}{x + \frac{1}{x^2}}$ by $\dfrac{1 - \frac{1}{x} + \frac{1}{x^2}}{1 + x + \frac{1}{x}}$.

3. Divide $\frac{x^2}{yz} - \frac{2x}{m} + \frac{xz}{ny} + \frac{yz}{m^2} - \frac{z^2}{mn}$ by $\frac{x}{y} - \frac{z}{m}$.

4. Multiply $\frac{15x^2 + 11xy - 14y^2}{21x^2 + xy - 10y^2}$ by $\frac{14x^2 - 3xy - 2y^2}{10x^2 + 9xy - 7y^2}$.

5. Simplify $\frac{1}{6}\left(\frac{1}{a-1} - \frac{1}{a+1} + \frac{a-2}{a^2-a+1} - \frac{a+2}{a^2+a+1}\right)$.

6. Reduce to lowest terms $\frac{a^3 - 2a^2 - 5a - 12}{a^4 - 5a^3 - 11a^2 - 5a + 4}$.

B. 1. Multiply $3 + \frac{53 - y}{y^2 - 4y - 21}$ by $2 - \frac{2y + 22}{y^2 - 3y - 10}$.

2. Multiply $1 + \frac{a}{2} + \frac{a^2}{3} + \frac{a^3}{4}$ by $1 - \frac{a}{2} + \frac{a^2}{3} - \frac{a^3}{4}$.

3. Divide the difference between $\frac{9}{16}x^4 + \frac{8}{3}x + 4$ and $x^3 + \frac{23}{9}x^2$ by the difference between $\frac{3}{4}x^2$ and $2(\frac{1}{3}x + 1)$.

4. Multiply $\frac{5a^2 - 10a - 75}{2a^2 - 2a - 40}$ by $\frac{4a^2 + 4a - 48}{5a^2 + 5a - 30}$.

5. Add together

$$\frac{1}{x^2 + 3x + 2},\quad \frac{2x}{x^2 + 4x + 3},\quad \text{and}\quad \frac{1}{x^2 + 5x + 6}.$$

6. If $m = n + \sqrt{n^2 - 1}$, show that $m + \frac{1}{m} = 2n$.

C. 1. Multiply $m^2 - 2mn + 4n^2 - \frac{6n^3}{m+2n}$ by $1 + \frac{6n^3}{m^3 + 2n^3}$.

2. Divide the product of $3 - 2a$ and $\frac{1}{4} - \frac{a}{6}$ by $\frac{3}{2a^3} + \frac{2}{3} - \frac{2}{a}$.

3. Multiply $\frac{6x^2 - xy - 12y^2}{6x^2 + 23xy + 20y^2}$ by $\frac{6x^2 + 7xy - 20y^2}{12x^2 - 16xy - 3y^2}$.

4. Simplify $\frac{x+1}{2(x^2+1)} + \frac{1}{6(x+1)} - \frac{2x-1}{3(x^2-x+1)}$.

5. Multiply $\left(\frac{a^2}{a^2-b^2}+\frac{b^2}{a^2+b^2}\right)$ by $\frac{(a^2-b^2)^2}{(a^2-b^2)^2+(a^2+b^2)^2}$.

6. If $x=1$, $y=-2$, $z=3$, find the value of
$(2x+3y+2z)^3-8y\sqrt[3]{5x^2-4y^3-2z^3-10}$.

Exercise 67.

A. 1. Find the value of $\frac{1}{a^2+b^2}\cdot\frac{1}{x^2-a^2}$
$-\frac{x}{(a^2+b^2)(x^2+b^2)(x+a)}-\frac{1}{(x^2-a^2)(x^2+b^2)}$.

2. Divide $\frac{\frac{y}{x^2}+\frac{1}{y}-\frac{1}{x}}{y-x}$ by $\frac{\frac{1}{x^3}-\frac{1}{y^3}}{\frac{1}{x^2}-\frac{1}{y^2}}$.

3. Simplify $b^2\left(\frac{1}{a-\sqrt{a^2-b^2}}-\frac{1}{a+\sqrt{a^2-b^2}}\right)$.

4. Resolve into factors $(x^2-y^2)z+(y^2-z^2)x+(z^2-x^2)y$.

5 If $z^2=x^2+y^2$ show that
$(x+y+z)(x+y-z)(x+z-y)(y+z-x)=4x^2y^2$.

6. Show that $4x^2-4y^2$ is a factor of
$(x^2+xy-y^2)^2-(x^2-xy-y^2)^2$.

B. 1. Resolve into factors $a^4-(b+c)a^3+(b+c)abc-b^2c^2$.

2. Simplify $\frac{4a^3}{a^4+b^4}-\frac{8a^7}{a^8-b^8}+\frac{2a}{a^2+b^2}+\frac{1}{a+b}$

3. Reduce to lowest terms $\frac{xy(a^2+b^2)+ab(x^2+y^2)}{xy(a^2-b^2)+ab(x^2-y^2)}$.

4. If $y=x-1$, and $z=x+2$, show that

$$\frac{x^2y^2z^2-x^2y^2-z^2+1}{x^2yz-\frac{z}{y}+y\left(x^2-\frac{1}{y^2}\right)}+1=x^2.$$

5. Divide $\dfrac{(a-b)(a^2-ab+b^2)}{a^3+b^3}$ by $\dfrac{a^3-b^3}{(a+b)(a^2+ab+b^2)}$.

6. Simplify the fraction $\dfrac{1}{a+1+\dfrac{1}{a-1+\dfrac{1}{a+1}}}$

C. 1. Resolve into factors $15(a+b)^2+14(a+b)(x+y) - 8(x+y)^2$.

2. Simplify $\left(a+b-\dfrac{a^3}{a^2-ab+b^2}\right)\div\left(a+b-\dfrac{b^3}{a^2-ab+b^2}\right)$.

3. Reduce to lowest terms $\dfrac{(a+b+c)^2-(a-b-c)^2}{2a(b^2+2bc+c^2)}$.

4. Add together $\dfrac{a+c}{(a-b)(x-a)}+\dfrac{b+c}{(b-a)(x-b)}+\dfrac{x+c}{(x-a)(x-b)}$.

5. Divide $\left(\dfrac{1+a}{1-a}+\dfrac{4a}{1+a^2}+\dfrac{8a}{1-a^4}-\dfrac{1-a}{1+a}\right)$ by $\left(\dfrac{1+a^2}{1-a^2}+\dfrac{4a^2}{1+a^4}-\dfrac{1-a^2}{1+a^2}\right)$.

6. If $a=2$, $b=3$, $c=4$, $x=2\frac{1}{2}$, find the value of $\dfrac{bc(x+a)}{(a-b)(a-c)}+\dfrac{ac(x+b)}{(b-a)(b-c)}+\dfrac{ab(x+c)}{(c-a)(c-b)}$.

Exercise 68.

A. 1. If $a=\frac{1}{3}$, $b=0$, $c=-\frac{1}{3}$, $d=-1$, find the value of

$$\frac{cd-ab}{\dfrac{a^2}{c^2}-\dfrac{a}{c}}-\frac{a-c}{a^2+c^2}.$$

2. Divide $1+2a$ by $1-3a$ to four terms of the quotient.

3. Resolve into factors $24a^2y^2 - 14ay^3 - 3y^4$.

4. Simplify $\frac{5a}{b}\left(4 - \frac{2a}{b} + \frac{3b}{a}\right) - \frac{2a^2}{b^2}\left(\frac{13b}{a} - 5\right) - 9$.

5. Find the L. C. M. of $4a^4 + 4a^3 - 8a^2$, $6a^5 - 12a^4 - 18a^3$, and $3a^2 - 3a - 18$.

6. If $A = x + y$, $B = x - y$, find the value of
$$\frac{A+B}{A-B} - \frac{A-B}{A+B}.$$

B. 1. If $a = 3$, find the value of $2a^{a+1} - (2a)^{a-1} - 3(a-1)^a$.

2. Divide $1 - 2y$ by $1 - y + 2y^2$ to 5 terms of the quotient.

3. Simplify.
$$\frac{ab}{a^2+b^2}\left\{\frac{a+b}{a-b} + \frac{a^3+b^3}{a^3-b^3}\right\} \div \left\{\frac{a+b}{a-b} - \frac{a^3+b^3}{a^3-b^3}\right\}.$$

4. Find the G. C. M. of $6x^3 - 19x^2 - 16x - 3$ and $2x^3 - 11x^2 + 11x + 6$.

5. Multiply $\frac{1}{2}x + \frac{1}{4}y + \frac{1}{8}z$ by $\frac{1}{2}x - \frac{1}{4}y + \frac{1}{8}z$.

6. Resolve into factors $75x^2 - 35x^3 - 10x^4$.

C. 1. Divide $6xyz\left(1 - \frac{a}{x} - \frac{b}{2y} - \frac{c}{3z}\right) + abc\left(\frac{x}{a} + \frac{2y}{b} + \frac{3z}{c} - 1\right)$ by the product of $x - a$ and $2y - b$.

2. Resolve into factors $abx^2 - (a^2 - b^2)x - ab$.

3. Simplify
$$\left(y - \frac{a^2 - xy}{y - x}\right)x + \left(\frac{a^2 - xy}{y - x}\right) + \left(\frac{a^2 - xy}{y - x}\right)^2.$$

4. Find the value of $\frac{x^4}{a^2b^2} + \frac{(x^2-a^2)^2}{a^2(a^2-b^2)} - \frac{(x^2-b^2)^2}{b^2(a^2-b^2)}$

5. Reduce to lowest terms $\frac{42a^4y - 7a^3y^2 - 7a^2y^3}{28a^3y^3 + 42a^2y^4 - 28ay^5}$.

6. If $x = \frac{1}{4}$, $y = \frac{1}{5}$, find the value of
$$\frac{x}{y} - \sqrt{\frac{1+x}{1-y}} + \sqrt{\frac{5x^2}{y^3}}.$$

Exercise 69.

A. 1. Resolve $ax^2 + (a + b)x + b$ into two factors.

2. Divide $5(3x^4 + 2) - 9x(1 - 3x) - 13x^3 + 6x^2$ by $5x^2 + 2 - x$.

3. Simplify $\left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right) + \left(x - \frac{1}{x}\right)\left(y - \frac{1}{y}\right)$.

4. If $a = -3$, $b = 5$, $x = -1$, find the value of $3a\sqrt{3ab + 46} - 2bx\sqrt[3]{a^3 - 5b + 12x}$.

5. If $x = \frac{a+1}{ab+1}$, $y = \frac{a(b+1)}{ab+1}$ find the value of $\frac{x+y-1}{x-y+1}$.

6. Find the value of $\frac{a^2+a}{a-1} \times \frac{a^3-1}{a^4+a} \div \frac{a^3+a^2+a}{a^2-a+1}$.

B. 1. Divide $y^3 - y^2 + y - 2 - \frac{2}{y^2} - \frac{2}{y^3}$ by $y^2 + 2 + \frac{2}{y^2}$.

2. Find the value of $\frac{a+b+c}{abc}(a^2 + b^2 + c^2) -$
$$\left(\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b}\right).$$

3. Resolve into factors $ab(x^2 - y^2) - (a^2 - b^2)xy$.

4. Simplify $\frac{a^2 - 6ab + 9b^2}{a^2 - 4ab + 4b^2} \div \left\{\frac{a^2 - 9b^2}{a^2 - 4b^2} \div \frac{a^2 + ab - 6b^2}{a^2 - ab - 6b^2}\right\}$.

5. If $2x^4 + 2x^3 - 11x^2 + 13x - 3$ and $2x^4 - 2x^3 - 5x^2 + 11x - 6$ have a common factor, resolve each into its factors.

6. Add together $\frac{4}{a^2 + a}$, $\frac{2a - 5}{a^2 - a + 1}$, and $\frac{11 - 2a^2}{a^3 + 1}$.

C. 1. By what must we divide $36y - 23y^2 + 12y^4 + 8$ to give quotient $3y^2 + 6y + 1$ and remainder $2y + 1$?

2. If $a = \frac{1}{2}$, $b = -\frac{1}{3}$, $x = -1$, $y = 0$, find the value of $abxy - \sqrt{\frac{1}{2}a - 2x} + \sqrt{3b - 9x}$.

3. Simplify $(2x - 3y)^2 \div \left\{ \frac{1}{6xy^3} - \frac{1}{2x^2y^2} + \frac{3}{8x^3y} \right\}$.

4. Resolve into factors $11x^2 - 54xy^2 + 63y^4$.

5. Simplify, without multiplying out,

$$\frac{(b + c - 2a)^2 - (c + a - 2b)^2}{(a + b + c)^2 - (a + b - 5c)^2}.$$

6. If $a = \frac{4x^2}{(1 + x^2)}$, find the value in terms of x of $\sqrt{1 - a} \div \{(1 + x)\sqrt{a} + \sqrt{1 - a}\}$.

Exercise 70.

A. 1. Which is the greater, and find the difference between $(2^2)^{2^2}$ and $2^{2^{2^2}}$.

2. Resolve into factors $4abx^2 - 2(a^2 + b^2)xy + aby^2$.

3. Show that the difference of the squares of two consecutive even numbers is 4 times the odd number between them.

4. Simplify

$$\frac{9b^2 - (4c - 2a)^2}{(2a + 3b)^2 - 16c^2} + \frac{16c^2 - (2a - 3b)^2}{(3b + 4c)^2 - 4a^2} + \frac{4a^2 - (3b - 4c)^2}{(4c + 2a)^2 - 9b^2}.$$

5. If $a = \cdot 1$, find the value of $a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \frac{a^5}{5}$ correct to 5 decimal places.

6. Divide $a + bx$ by $c - dx$ to four terms of the quotient.

B. 1. Simplify $2^{2^{2}} \div 2\,(2^2)^{2}$.

2. Resolve into factors $7x^3 + 96x^2 - 103x$.

3. Divide $x^4 - y^4$ by the difference between $(1 + x^3) + (1 + x)^2y + (1 + x)y^2 + y^3$ and $3x(x + 1) + y(y + 1) + 2xy + 1$.

4. Simplify $\dfrac{a^2-(b-c)^2}{(a+c)^2-b^2}+\dfrac{b^2-(c-a)^2}{(a+b)^2-c^2}+\dfrac{c^2-(a-b)^2}{(b+c)^2-a^2}$.

5. Find the difference between the cube of the sum of the squares a and b, and the square of the sum of their cubes.

6. Show that the cube of the difference of a fraction and its reciprocal is less than the difference of their cubes by 3 times the difference of the fraction and its reciprocal.

C. 1. Resolve into factors $ax^2 + (a - b)x - b$.

2. Simplify $\left(\dfrac{x}{y}+\dfrac{y}{x}\right)\left(\dfrac{a}{b}+\dfrac{b}{a}\right)-\left(\dfrac{x}{y}-\dfrac{y}{x}\right)\left(\dfrac{a}{b}-\dfrac{b}{a}\right)$.

3. Find both the G.C M. and L.C.M of $49a^2 - 4b^2$, $21a^2 - 20ab + 4b^2$, $49a^2 - 28ab + 4b^2$, and $21a^2 + 8ab - 4b^2$.

4. Simplify $\dfrac{(x-y)(y-z)+(y-z)(z-x)+(z-x)(x-y)}{x(z-x)+y(x-y)+z(y-z)}$.

5. Write at full length the four products of $(ax \pm b)$ by $(cx \pm d)$.

6. What number must be added to $a^3 + a^2 - 4(a + 3)$ that it may be exactly divisible by $a - 6$?

ALGEBRA FOR MIDDLE SCHOOLS.

ANSWERS TO EXAMPLES.

Chap. I.—INTERPRETATION OF ALGEBRAIC SIGNS.

Exercise 1.

	1.	2.	3.	4.
A.	20.	7008.	39.	256.
B.	36.	12480.	496.	30.
C.	159.	6400.	232.	118.

Exercise 2.

	1.	2.	3.	4.
A.	10.	18.	70.	78.
B.	1.	18.	328.	96.
C.	26.	26.	228.	60.

Exercise 3.

	1.	2.	3.	4.
A.	52.	—5.	9.	15.
B.	10.	36.	71.	26.
C.	35.	34.	66.	66.

Exercise 4.

	1.	2.	3.	4.
A.	5.	9.	12.	5.
B.	9.	7.	14.	4.
C.	4.	0.	6.	14.

Exercise 5.

	1.	2.	3.	4.
A.	$z + y - x$.	$6x + 7y - 9z$.	$3a + 4b = 5c$.	$ab = xy$.
B.	$3z + x - y$.	$6b - 7a + 3c$.	$5y - 2x = 4z$.	$pm = rs$.
C.	$by + ax$.	$9b - 4f + 3d$.	$3b - 10a = d - 2c$.	$2d + 3abc$.

Exercise 6.

A. **1.** Twice a diminished by 3 times b and increased by 4 times c. **2.** Twice x increased by y and also by p times a. **3.** Twice a diminished by the half of b, and increased by c. **4.** Three times the product of b and c diminished by four times the product of a and y is equal to five times the product of b and z.

B. **1.** Four times p increased by twice q and diminished by five times r. **2.** Three times y diminished by four times x and further diminished by twice z. **3.** Nine times a diminished by twice d and increased by half x. **4.** Four times the product of b and y diminished by twice the product of a and x is equal to nought.

C. **1.** c increased by four times d and diminished by twice b. **2.** Three times y increased by five times z and diminished by eight times x. **3.** One-half c increased by d and diminished by a is equal to nought. **4.** Five times the product of p and q diminished by twice the product of x and y.

Exercise 7.

	1.	2.	3.	4.
A.	$b - a$.	$x + 20$.	$7ax$.	$y + 12$.
B.	$20x$.	$n+2, n+4, n-4, n-2$.	A. $x+y$, B. $x-y$.	$m + 10$.
C.	$3ab$.	$x+1, x+3, x+5, x+7$.	$r - p$.	$30 - 3x$.

Chap. II.—ADDITION.

Exercise 8.

	1.	2.	3.	4.
A.	$18x + 12y$.	$14x - 10y$.	$6b - 2a$.	$5c + 7d$.
B.	$16a + 15b$.	$17m - 6n$.	$6y - 4x$.	$5b - 3a$.
C.	$11m + 11n$.	$18c - 9d$.	$3y - 4x$.	$2m - 15n$.

Exercise 9.

	1.	2.	3.	4.
A.	$-2a + b$.	$4x - 5y$.	$8c - 18d$.	$-6p + 4r$.
B.	$x - 8y$.	$12p + 10s$.	$-7a + 8b$.	$2x - 18a$.
C.	$8p + 9c$.	$-5a + b$.	$-14x - 3y$.	$18x - 2y$.

Exercise 10.

A. 1. $-a^2 + 7b^2 - 2c^2 - 16d^2$. **2.** $4xy + 26yz - 10xz$. **3.** $19abc - 22bcd + acd$. **4.** $2x^2y + xy^2 + 7x^2y^2$.

B. 1. $x^2 - 16y^2 + 4z^2$. **2.** $-17pq - 16pr + 5qr - 5qs$. **3.** $6xyz - 2x^2y^2z^2 - 4xy^2z - 9xyz^2 - 8x^2yz$. **4.** $5abcd - 8ab^2c^2d - a^2bcd^2$.

C. 1. $9mnp - 25m^2n^2p^2 - 7m^3n^3p^3$. **2.** $-3abp - 3a^2b^2 + 17a^2q^2 - 9b^2q^2$. **3.** $11axyz - 7bx^2yz - 2cxyz^2$. **4.** $-9mnp^2 - 7m^2np + 7mn^2p + 3m^2n^2p^2$.

Chap. III.—SUBTRACTION.

Exercise 11.

	1.	2.	3.	4.
A.	$-2b$.	$x - 2y - 2z$.	$b + c$.	$-2b - c$.
B.	$2y$.	$c - 3d - 2f$.	$y + z$.	$5a - 2c$.
C.	$-d$.	$p + 4q - 2r$.	$9p + 9r + 3s$.	$-2p - q + 2s$.

Exercise 12.

A. **1.** $3xy + 12xz - 9yz$. **2.** $2p - 9q + 11r$. **3.** $4b - 17c$. **4.** $5pq + pr - 21qr$.

B. **1.** $2a + 5b - 2c$. **2.** $3p - 9q + 2r$. **3.** $-2xy + 11xz - 13yz$. **4.** $2xy + 4xz - yz$.

C. **1.** $p + 4q - 3r$. **2.** $3y - 3z$. **3.** $ab - 13ac + 21bc$. **4.** $3a^2 + 3b^2 - c^2$.

Chap. IV.—EXTENDED USE OF + AND —

Exercise 13.

A. **1.** 8 cows, 8 sheep and 9 horses. **2.** 22 men, 20 women and 18 boys. **3.** 26 pounds, 2 shillings and 10 pence.

B. **1.** 29 chairs, 9 cots, and 15 tables. **2.** 232 rice, 116 ragi, and 90 cholum. **3.** 1920 infantry, 764 cavalry, and 540 artillery.

C. **1.** 24 cl. I, 32 cl. II, and 33 cl. III. **2.** 64 sovs., 29 crs., and 31 shillings. **3.** 11 first, 48 second, and 575 third class.

Chap. V.—THE USE OF BRACKETS.

Exercise 14.

	1.	2.	3.	4.
A.	$x + 4y + 2z$.	$a + 5c$.	$8x - 9y$.	$11m - 12n$.
B.	$a + b + 8c$.	$m - 3n + 10p$.	$32a - 41b$.	$21a - 19b$.
C.	$c + 5d + 11c$.	$x + 2y - 7z$.	$28a - 13b$.	$27x - 5y$.

Exercise 15.

	1	2.	3.	4.
A.	3.	$4x$.	$11x - 34y$.	$-36m + 44n$.
B.	6.	$-a$.	$11a - 16b$.	$-43x + 12y$.
C.	9.	$5m$.	$6m + 9n$.	$-43a + 94b$.

Chap. VI—MULTIPLICATION.

Exercise 16.

A. **1.** $6abx^5$; $-12x^5y^7$; $24a^5b^4x^4y^5$. **2.** $24p^4q^5-18p^5q^4+30p^4q^4$. **3.** $-9a^3b^2c^2+6a^2b^3c^2+12a^2b^2c^3+18a^3b^3c^3$. **4.** $-24x^4y^2z^5-18x^4y^3z^4+42x^5y^2z^4-108x^4y^2z^4$.

B. **1.** $12p^3x^6$; $-2x^7y^6$; $40a^5b^4x^5$. **2.** $-6a^2b^2c^2x^3y^2z+9a^2bc^4xy^3z-12a^3bc^2xy^2z^4$. **3.** $-35a^5x^3y^3+21a^4x^4y^2+14a^4x^3y^2-28a^5x^4y^3$. **4.** $-8p^4q^3x^4y^2z^2+20p^5q^2x^2y^4z-12p^5q^3x^3y^2z^3-4p^4q^2xy^5z^4$

C. **1.** $45x^4y^6$; $-12p^3x^5y^4$; $28a^5b^3x^3y^6$. **2.** $-12a^2bx^4yz^2+14ab^3x^2y^4z^3+6abc^3x^2yz^5$. **3.** $10p^3q^3x^3-15p^4q^2x^4z^2-20p^5q^3x^5$. **4.** $-24a^5b^4x^5y^4+30a^4b^5x^7y^3+12a^4b^4x^7y^4+18a^5b^5x^6y^5$.

Exercise 17.

A. **1.** $6x^2+23x+21$, and $20x^2+18x-18$. **2.** $8a^3+6a^2+a+15$. **3.** $15x^3+2x^2+37x+36$. **4.** $2x^4-8x^3+14x^2-14x+4$.

B. **1.** $12x^2+38x+30$, and $14x^2-22x-12$. **2.** $15y^3-16y^2+25y-14$. **3.** $14y^3+9y^2-2y+24$. **4.** $8p^4-24p^3+26p^2-16p+6$.

C. **1.** $6y^2+31y+40$, and $20y^2+2y-42$. **2.** $12x^3-17x^2+42x-27$. **3.** $12p^3-40p^2+10p+28$. **4.** $6a^4-29a^3+18a^2+39a-14$.

Exercise 18.

A. **1.** x^4-x^2+2x-1. **2.** $18x^4-45x^3+82x^2-67x+40$. **3.** $y^6-3y^5-3y^4+13y^3-6y^2-6y+4$. **4.** $x^4+x^3y-9x^2y^2-23xy^3-10y^4$.

B. **1.** $p^2+pq+2pr-2q^2+7qr-3r^2$. **2.** $3a^2+8ab+4b^2+10ac+8bc+3c^2$. **3.** $m^5-5m^3+5m^2-1$. **4.** $4x^2y^2+6x^4-5x^3y-4y^4+11xy^3$.

C. **1.** $4x^5-x^2+4x$. **2.** $3p^4+14p^3+9p+2$. **3.** $a^5-4a^4b+4a^3b^2+4a^2b^3-17ab^4-12b^5$. **4.** $6y^4+5y^3+6y^2-17y+6$.

Exercise 19.

A. 1. $a^5 + 4a^4 + 11a^3 - 15a^2 - 56a + 15$. **2.** $6y^4 + 5y^3 + 6y^2 - 17y + 6$. **3.** $6y^5 + 5y^4 - 25y^3 + 31y^2 - 13y + 2$. **4.** $a^5 + a^3b^2 + a^2b^3 + b^5$.

B. 1. $3a^4 - 23a^3b + 27a^2b^2 - 34ab^3 + 12b^4$. **2.** $1 + 6y + 17y^2 + 34y^3 + 52y^4 + 40y^5$. **3.** $16x^5 + 16x^4y - 30x^3y^2 + 55x^2y^3 - xy^4 - 36y^5$. **4.** $y^5 - (a^2 + 2b)y^3 + cy^2 + (b^2 - ac)y - bc$.

C. 1. $6y^5 - 17y^4 + 21y^3 - 17y^2 + 9y - 2$. **2.** $36 + 73a + a^2 - 5a^3 - 15a^4 - 18a^5$. **3.** $27x^7 + 51x^6 + 4x^5 - 58x^4 + 55x^3 - 13x^2 + 38x - 24$. **4.** $z^5 - 2az^4 + (a^2 + 2b)z^3 - (2ab + c)z^2 + (ac + b^2)z - bc$.

Exercise 20.

A. 1. $a^3 + (b+c+d)a^2 + (bc+bd+cd)a + bcd$. **2.** $30x^3 + 178x^2y + 328xy^2 + 192y^3$. **3.** $216a^4 - 486a^3b - 528a^2b^2 + 1482ab^3 - 420b^4$. **4.** $a^5 - a^4x + a^3x^2 - a^2x^3 + ax^4 - x^5$.

B. 1. $x^3 + (p + y + z)x^2 + (yz + py + pz)x + pyz$. **2.** $120a^3 + 388a^2c + 368ac^2 + 84c^3$. **3.** $432x^4 + 36x^3y - 1608x^2y^2 - 564xy^3 + 360y^4$. **4** $c^2x^2(a^4 + y^4) - a^2y^2(c^4 + x^4)$.

C. 1. $m^3 + (n + p + q)m^2 + (np + nq + pq)m + npq$. **2.** $90x^3 - 327x^2y + 309xy^2 - 84y^3$. **3.** $120a^4 - 286a^3b + 21a^2b^2 + 414ab^3 - 216b^4$. **4.** $360 + 78x - 35x^2 - 4x^3 + x^4$.

Exercise 21.

A 1. $\frac{1}{6}x^3 - \frac{5}{36}x^2y + \frac{1}{6}xy^2 + \frac{2}{9}y^3$. **2.** $\frac{1}{4}p^3 + \frac{1}{72}p - \frac{1}{12}$. **3.** $\frac{1}{3}m^3 - \frac{25}{72}m^2n + \frac{1}{3}mn^2 - \frac{1}{6}n^3$. **4.** $\frac{1}{2}a^3 + \frac{3}{24}a^2x + \frac{7}{36}ax^2 + \frac{1}{3}x^2$.

B 1. $\frac{2}{3}a^3 - \frac{17}{12}a^2b + \frac{35}{24}ab^2 - \frac{5}{4}b^3$. **2.** $\frac{1}{6}y^3 - \frac{7}{8}xy^2 + \frac{35}{24}x^2y - \frac{3}{4}x^3$. **3.** $\frac{1}{3}p^3 - \frac{11}{36}p^2q + \frac{11}{36}pq^2 - \frac{1}{6}q^3$. **4.** $\frac{1}{8}x^3 - \frac{9}{32}x^2 + \frac{1}{8}x - \frac{1}{8}$.

C. 1. $\frac{1}{2}m^3 - \frac{13}{12}m^2n + \frac{1}{3}mn^2 - \frac{1}{3}n^3$. **2.** $\frac{1}{6}x^3 - \frac{11}{8}x^2y + \frac{31}{12}xy^2 - \frac{3}{4}y^3$. **3.** $\frac{5}{8}a^3 - \frac{1}{24}a^2b - \frac{3}{8}ab^2 + \frac{1}{12}b^3$. **4.** $\frac{3}{8}p^3 - \frac{5}{4}p^2q + \frac{5}{4}pq^2 - \frac{1}{2}q^3$.

Chap VII.—DIVISION.

Exercise 22.

A. 1. $3a^2x^2$; $-2ab^3y^2$. **2.** $3x^4-4x^3+2x^2$. **3.** $3x^2+2x-6$. **4.** $3ax+6ay+7az$.

B. 1. $5ac^2x^3$; $-3b^3cy^4$. **2.** $4y^3-6y^2-7y$. **3.** $7y^3-4y^2+9y$. **4.** $4a^2x^2+8axy+4y^2$.

C. 1. $3a^4b^2x^3$; $-2a^2b^2x^3$. **2.** $8x^3-7x^2+6x$. **3.** $4m^3-3m^2+4m$. **4.** $2a^2x^2-4axz+2z^2$.

Exercise 23.

A. 1. x^3+2x^2+2x+1. **2.** $x^3+x^2y+xy^2+y^3$. **3.** $3x^4-4x^3+5x^2-6$. **4.** $\frac{1}{2}x^2-\frac{2}{3}x-\frac{3}{4}$.

B. 1. $3x^3+7x^2-5x+1$. **2.** $a^3+a^2b+ab^2+b^3$. **3.** $4x^5-6x^3+5x^2-2x+1$. **4.** $\frac{1}{3}a^2+\frac{2}{3}ax+\frac{2}{3}x^2$.

C. 1. $3x^2-2xy+4y^2$. **2.** $p^3+p^2q+pq^2+q^3$. **3.** $5x^5-3x^4+7x^3-4x+9$. **4.** $\frac{3}{2}x^2-ax-\frac{2}{3}a^2$.

Exercise 24.

	1.	2.	3.	4.
A.	$x+9$.	$ac+16$.	$2+2a+a^2$.	$-\frac{1}{72}a^2x^2+\frac{1}{3}x^4$.
B.	$2x+10$.	$13-ac$.	$2-x$.	$-\frac{1}{3}a-\frac{1}{9}$.
C.	$2xy+8$.	$x+1$.	$x-4$.	$\frac{1}{4}-xy$.

Exercise 25.

A. 1. $\frac{1}{a}-\frac{x}{a^2}+\frac{x^2}{a^3}-\frac{x^3}{a^4}+$ &c. **2.** $\frac{1}{x}-\frac{1}{x^3}+\frac{1}{x^5}-\frac{1}{x^7}+$ &c.

3. $1+2x+x^2-x^3(1+2x+x^2)+$ &c. **4.** $\frac{1}{b}-\frac{1}{ab^2}+\frac{1}{a^2b^3}-\frac{1}{a^3b^4}+$ &c.

B. 1. $\frac{3}{x}-\frac{3c}{x^2}+\frac{3c^2}{x^3}-\frac{3c^3}{x^4}+$ &c. **2.** $\frac{1}{x}-\frac{1}{x^3}+\frac{1}{x^5}-\frac{1}{x^7}+$ &c. **3.** $\frac{1}{x}-\frac{1}{x^4}+\frac{1}{x^7}-\frac{1}{x^{10}}+$ &c. **4.** $1+5y+15y^2+54y^2+$ &c.

C. 1. $\frac{1}{a} - \frac{x^2}{a^3} + \frac{x^4}{a^5} - \frac{x^6}{a^7} + \&c.$ **2.** $\frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^5} + \frac{1}{x^7} + \&c.$ **3.** $-\frac{1}{y} - \frac{1}{y^5} - \frac{1}{y^9} - \frac{1}{y^{13}} - \&c.$ **4.** $1 + x + 2x^2 + 2x^3 + \&c.$

Chap. VIII.—MULTIPLICATION AT SIGHT.

Exercise 26.

A. 1. $x^2 + 20x + 99$; and $x^2 - 2x - 24$. **2.** $x^3 + (3a + 2b + 2c)x^2 + (6ab + 6ac + 4bc)x + 12abc$. **3.** $y^3 - 5my^2 - 2m^2y + 24m^3$. **4.** $x^3 - 24x^2 + 191x - 504$.

B 1. $x^2 + 14x + 48$; and $x^2 - 7x - 30$. **2.** $x^3 + 12mx^2 + 44m^2x + 48m^3$. **3.** $y^3 - 8py^2 - 5p^2y + 84p^3$. **4.** $s^3 - 2(a + b + c)s^2 + 4(ab + ca + bc)s - 8abc$.

C. 1. $y^2 + 12y + 32$; and $y^2 - 5y - 66$. **2.** $p^3 + (2q + 3v + 4s)p^2 + (6qr + 8qs + 12rs)p + 24qrs$. **3.** $x^3 - 12cx^2 + 39c^2x + 28c^3$. **4.** $a^3 - 6a^2 - 37a + 210$.

Exercise 27.

A. 1. $9x^2 + 25y^2 + 30xy$; $16x^2 + 49y^2 - 56xy$. **2.** $25y^2 - 18$; $4x^2 - 9z^2$. **3.** $1 + 4x + 10x^2 + 12x^3 + 9x^4$; and $1 - 4x - 2x^2 + 12x^3 + 9x^4$. **4.** $x^4 + y^4 + x^2y^2$; $x^4 - x^2 - 2x - 1$.

B. 1. $16a^2 + 9b^2 + 24ab$; $9m^2 + 25n^2 - 30mn$. **2.** $36a^2 - 100b^2$; $9y^2 - 16c^2$. **3.** $1 + 2y + 5y^2 + 4y^3 + 4y^4$; and $1 - 2y - 3y^2 + 4y^3 + 4y^4$. **4.** $a^4 + b^4 + a^2b^2$; and $y^4 - 6y^2 + 1$.

C. 1. $25y^2 + 9x^2 + 30xy$; $64a^2 + 16b^2 - 64ab$. **2.** $9x^2 - 16y^2$; $25p^2 - 4q^2$. **3.** $a^4 + 6a^3 + 13a^2 + 12a + 4$; and $9x^4 - 18x^3 + 3x^2 + 6x + 1$. **4.** $a^4 + a^2 + 1$; and $9b^4 - 15b^2 + 1$

Exercise 28.

A. 1. $8a^3 + 36x^2y + 54xy^2 + 27y^3$; and $27x^3 - 54x^2y + 36xy^2 - 8y^3$. **2.** $27m^3 + 64n^3$; and $m^3 + 1$. **3.** $a^6 - a^3b^3$ and $p^3 - q^3$. **4.** $x^6 + y^6$; and $x^9 - y^9$.

B. 1. $64a^3 + 96a^2b + 48ab + 8b^3$; and $8x^3 - 36x^2c + 34xc^2 - 27c^3$. **2.** $125x^3 + 27y^3$; and $a^3 - 1$. **3.** $m^6 - p^3$; and $27c^3 - 8d^3$. **4.** $a^6 + 64b^6$; and $1 - p^9$.

C. 1. $125a^3 + 225a^2x + 135ax^2 + 27x^3$; and $27y^3 - 108y^2 + 144y - 64$. **2.** $8p^3 + 125q^3$; and $27p^3 - 1$. **3.** $64x^6 - 27$; and $27y^6 + 8$. **4.** $8 + y^6$; and $m^6 - 1$.

Chap. IX.—DIVISION AT SIGHT.

Exercise 29.

A. 1. $3a + 2b$; $4x^2 - 3y$. **2.** $p + 2$; and $\frac{1}{4}x^2 + \frac{1}{6}x + \frac{1}{9}$. **3.** $x^8 - x^6y^2 + x^4y^4 - x^2y^6 + y^8$; and $\dfrac{1}{y^3} + \dfrac{1}{y^2z} + \dfrac{1}{yz^2} + \dfrac{1}{z^3}$. **4.** $27m^3 + 36m^2n + 48mn^2 + 64n^3$.

B. 1. $9x - 6y^2$; $5m^2 + 7n^3$. **2.** $a^6 + 3a^4 + 9a^2 + 27$; and $\dfrac{1}{p^2} - \dfrac{1}{pq} + \dfrac{1}{q^2}$. **3.** $m^9 + m^6p^2 + m^3p^4 + p^6$; $\dfrac{1}{a^4} - \dfrac{1}{a^2b^3} + \dfrac{1}{b^6}$. **4.** $8a^3x^3 + 12a^2x^2by + 18axb^2y^2 + 27b^3y^3$.

C. 1. $2m - 3n^2$; $6a^3 + 5b^2$. **2.** $x^3 + 2x^2y + 4xy^2 + 8y^3$; and $p + 2q$. **3.** $m^{12} + 3m^8n^4 + 9m^4n^8 + 27n^{12}$; and $a^{20} - a^{15}b^5 + a^{10}b^{10} - a^5b^{15} + b^{20}$. **4.** $p^9 + p^6 + p^3 + 1$; and $x^6 - x^3y^4 + y^8$.

Exercise 30.

A. 1. $4x^2 - 6xy + 9y^2$. **2.** $a^2 + b^2 + 1 - ab - a -$ **3.** $m^4 + m^2 + 1$. **4.** $m^4 + m^2 + 1$.

B. 1. $a^2 + 2ab + 4b^2$. **2.** $x^2 + 4y^2 + 1 - 2xy - x - 2y$. **3.** $4a^2 - 2a + 1$. **4.** $x^2 + m^2 + a^2b^2 - mx - abx - abm$.

C. 1. $16p^2 + 12pq + 9q^2$. **2.** $x^2 + y^2 + 4 - xy - 2x - 2y$. **3.** $9a^2 + 3a + 1$. **4.** $x^2 + y^2 + 9z^2 + xy - 3xz + 3yz$.

Chap. X.—FACTORS.

Exercise 31.

A. 1. $3x(x - 5)$. **2.** $4m(m^3 - 4m - 2)$. **3.** $3a^2(3a^3 - a + 8)$. **4.** $5a^3x^3(a^3 - 3x + 5ax^2)$.

B 1. $4a^3(1 - 3a^2)$. **2.** $9x^2(1 - 2x^2 + 3x^4)$. **3.** $5p^4(2p^4 + 3 - 4p)$. **4.** $6a^3x^2(a^3 + 2x^2 - 3ax^4)$.

C. 1. $6b^5(1 - 3b^2)$. **2.** $3a^4(a - 4 + 7a^2)$. **3.** $7a^4b^2(a^4 - 3b^2 - 5a^2b^4)$. **4.** $5m^3n^4(1 - 2mn + 3m^2n^2)$.

Exercise 32.

A. 1. $(a + 2b)(c + a)$. **2.** $(x - m)(x + 4)$. **3.** $(y + 1)(y^3 + 2)$. **4.** $(p^2 + q^2)(x^2 - b)$.

B. 1. $(2a + b)(x + y)$. **2.** $(x - 5)(x + a)$. **3.** $(3y + 5)(y^2 + 1)$. **4.** $(2x - 3y)(bx - ay)$.

C. 1. $(3a + c)(m + n)$. **2.** $(4 + b)(a + b)$. **3.** $(2y - 1)(y^3 + 2)$. **4.** $(b + ac)(xy - z)$.

Exercise 33.

A. 1. $(m+3)(m+5)$. **2.** $(x+4a)(x+6a)$. **3.** $(y^2 - 5)(y^2 - 6)$. **4.** $(a^2 + 3b)(a^2 + 2b)$.

B. 1. $(p + 6)(p + 2)$. **2.** $(n - 5)(n - 10)$. **3.** $(x^2 + 7a)(x^2 + 4a)$. **4.** $(c + 8)(c + 4)$.

C. 1. $(x+8)(x+6)$. **2.** $(m^2 - 4)(m^2 - 1)$. **3.** $(a - 10b)(a - 3b)$. **4.** $(ab - 5c)(ab - 2c)$.

Exercise 34.

A. 1. $(x^2+7)(x^2+8)$. **2.** $(x - 11)(x+4)$. **3.** $(x - 6a)(x + 7a)$. **4.** $(p - 3m)(p + 5m)$.

B. 1. $(a - 9)(a - 6)$. **2.** $(y^2+10)(y^2 - 3)$. **3.** $(y - 4ab)(y + 2ab)$. **4.** $(m - 2nx)(m + 3nx)$.

C. 1. $(x - 3)(x - 8)$. **2.** $(ay+7)(ay - 4)$. **3.** $(p+3qx)(p - 2qx)$. **4.** $(mx - 6y)(mx + 9y)$.

Exercise 35.

A. **1.** $(a+1)(2a+1)$. **2.** $(m+3)(3m+1)$. **3.** $(x+2)(3x-1)$. **4.** $(x-7)(4x+6)$.

B. **1.** $(a+1)(3a+2)$. **2.** $(m+4)(2m+1)$. **3.** $(x-4)(2x+3)$. **4.** $(x-8)(5x+6)$.

C. **1.** $(a+2)(2a+1)$. **2.** $(m+2)(3m+2)$. **3.** $(x-5)(4x+1)$. **4.** $(x-5)(3x+7)$

Exercise 36.

A. **1.** $(4x+6)(3x+4)$. **2.** $(5a-9)(3x-4)$. **3.** $(5x+6y)(4x-3y)$. **4.** $(7a+2b)(2a-7b)$.

B. **1.** $(2a+5)(3a+2)$. **2.** $(4m-6)(3m-4)$. **3.** $(10a-3b)(2a+7b)$. **4.** $(3x-5y)(2x+5y)$.

C. **1.** $(4y-7)(3y-4)$. **2.** $(6x+5)(2x+3)$. **3.** $(7x-3a)(5x+4a)$. **4.** $(11x-4z)(3y+8z)$.

Exercise 37.

A. **1.** $(3x-5y)(3x+5y)$. **2.** $(11a-3b)(11a+3b)$ **3.** $\{\overline{a+b}+\overline{x+y}\}\times\{\overline{a+b}-\overline{x+y}\}$. **4.** $\{2a+\overline{y+z}\}\times\{2a-\overline{y+z}\}$.

B. **1.** $(6p-7q)(6p+7q)$. **2.** $(3a^2-4b^2)(3a^2+4b^2)$. **3.** $\{\overline{m+2n}+x\}\{\overline{m+2n}-x\}$. **4.** $\{\overline{2a-3b}+\overline{4c-d}\}\times\{\overline{2a-3b}-\overline{4c-d}\}$.

C. **1.** $(2a-7b^2)(2a+7b^2)$. **2.** $(16y^2+5x^2)(16y^2-5x^2)$. **3.** $\{\overline{1+x}-\overline{2a+3b}\}\times\{\overline{1+x}+\overline{2a+3b}\}$. **4.** $\{2a+\overline{y-z}\}\times\{2a-\overline{y-z}\}$.

Exercise 38.

A. 1. $b(2a + b)$. **2.** $47b(b + 2c)$. **3.** $(5a + 4b)(a + 10b)$. **4.** $(11x + y)(x + 9y)$.

B. 1. $b(2a - b)$. **2.** $(8y + z)(2y + 3z)$. **3.** $(7x + y)(x + 5y)$. **4.** $(a - b)(19a - 7b + 6)$.

C. 1. $(b - c)(b + 7c)$. **2.** $5b(6a - 5b)$. **3.** $(11x + 3y)(3x + 9y)$. **4.** $(a + b)(5b - 5a - 2)$.

Exercise 39.

A. 1. $(2c - 3a)(4c^2 + 6ca + 9a^2)$. **2.** $(x - 6y)(x^2 + 6xy + 36y^2)$. **3.** $(4x^2 + 6xy + 9y^2)(4x^2 - 6xy + 9y^2)$. **4.** $(25m^2 + 5m + 1)(25m^2 - 5m + 1)$.

B. 1. $(7a + 8)(49a^2 - 56a + 64)$. **2.** $(10x - 1)(100x^2 + 10x + 1)$. **3.** $(9x^2 + 15xy + 25y^2)(9x^2 - 15xy + 25y^2)$. **4.** $(a^2b^2 - ab + 1) \times (a^2b^2 + ab + 1) \times (a^4b^4 - a^2b^2 + 1)$.

C. 1. $(a + 4b)(a^2 - 4ab + 16b^2)$. **2.** $(9xy - 2a)(81x^2y^2 + 18axy + 4a^2)$. **3.** $16(4x^2 - 2xy + y^2)(4x^2 + 2xy + y^2)$. **4.** $((m^4y^4 + m^2y^2 + 1) \times (m^4y^4 - m^2y^2 + 1) \times (m^8y^8 - m^4y^4 + 1)$.

Exercise 40.

A. 1. $20b(5a + b)(5a - b)$. **2.** $5y^2(y - 6)(y + 3)$. **3.** $4x^2(2x - 1)(3x + 1)$. **4.** $8m(m^2 - 3)(m - 4)$.

B. 1. $x^2(x - 2y)(x^2 + 2xy + 4y^2)$. **2.** $(y - z - x)(y - z + x)$. **3.** $5xy(3x - 2)(3y - 4)$. **4.** $3p(2p^2 - 5)(p - 2)$.

C. 1. $y^3(b - 4c)(b + 4c)$. **2.** $7a^2b(2b - a)(b - 2a)$. **3.** $2a^2b(5a - 3b)(2a + 3b)$. **4.** $4m(m^2 + 6)(m - 2)$.

Exercise 41.

A. 1. $(x-a)(x+b)(x-c)$. **2.** $(x-2)(x+3)(x-5)$. **3.** $ab(a+b)(a-b)^2$. **4.** $(x+y+z)(x+y-z) \times (x-y+z)(x-y-z)$.

B. 1. $(y+a)(y-b)(y-c)$. **2.** $(x+1)(x-7)(x+4)$. **3.** $6b^2(4a+3b)(a-2b)$. **4.** $2a^2b(a+b)$.

C. 1. $(z+a)(z+b)(z-c)$. **2.** $(y+2)(y-4)(y-6)$. **3.** $5xy(xy-1)(x^2y^2+xy+1)$. **4.** $2(x-z)(1-xz)$.

Chap. XI.—GREATEST COMMON MEASURES.

Exercise 42.

	1.	**2.**	**3.**	**4.**
A.	$17a^2x^2y^3$.	$3a^2bc^2$.	$7abx^2$.	$3x^2yz$.
B.	$13m^2n^3x^4$.	$7x^2yz^3$.	$9p^2qs^3$.	$4x^2yz^4$.
C.	$16xy^3z^5$.	$7b^2c^2d$.	$5ab^2c$.	$5a^3bx^2$.

Exercise 43.

	1.	**2.**	**3.**	**4.**
A.	$x+y$.	$a+b$.	$y(x+y)$.	$3x+2$.
B.	$a+b$.	$xy(x-y)$.	$m+2n$.	$2y-3$.
C.	$x+2y$.	$2a-3b$.	$y+2$.	$4a-1$.

Exercise 44.

	1.	**2.**	**3.**	**4.**
A.	$a+3$.	$2y+1$.	$a(b+c)$.	$x-5$.
B.	$2x+3$.	$a^2(3a+2)$.	$a-2b$.	$2y(y^2-3)$.
C.	$3y-2$.	$a(y+a)$.	$x-3y$.	$(x-1)(x-6)$.

Exercise 45.

	1.	2.	3.	4.
A.	$3a + 9$.	$4y + 12$.	$a^4 + a^3 - 4a^2 + a + 1$.	$3(x-y)$.
B.	$2y + 11$.	$9a - 3$.	$y^2 - 2y + 4$.	$3y^2 + x^2$.
C.	$4b + 7$.	$x + 8$.	$a^3 + a^2 - a - 1$.	$p + 3$.

Exercise 46.

	1.	2.	3.	4.
A.	$a^2 - 3a + 2$.	$b^2 - 5$.	$2y^2 - 7$.	$x + 7a$.
B.	$y^2 - 13y + 5$.	$3a^2 + 1$.	$y^2 - y - 1$.	$x^2 - 2x + 1$.
C.	$s^2 - 8$.	$2b^2 - 3$.	$3x^2 - xy - 2y^2$.	$3y - 1$.

Exercise 47.

	1.	2.	3.	4.
A.	$a - 1$.	$y^2 - 2y + 5$.	$3x - 2y$.	$3x^2 - 4$.
B.	$x + 1$.	$a - 2$.	$y + 6$.	$2x - 3$.
C	$a - 1$.	$x^2 + xy + y^2$.	$x + y$.	$x - 3$.

Chap. XII.—LEAST COMMON MULTIPLE.

Exercise 48.

	1.	2.	3.	4.
A.	$60a^3b^2c^3$.	$12x^3y^3$.	$102m^3n^2p^3$.	$1176a^3b^2c^3$.
B.	$30x^4y^3$.	$12a^2b^3$.	$174a^2bc^2$.	$156m^4n^4$.
C.	$12a^3b^4$.	$12x^4y^4$.	$180a^2b^3cd^2$.	$216x^3y^3z^4$.

Exercise 49.

A. **1.** $8x(x-2y)^2$. **2.** $120ab(a^2-b^2)$ **3.** $(m+7)(m-6)(m-5)$. **4.** $(a+2)(a+3)(5a+1)$.

B. **1.** $18ab(a^2-b^2)$. **2.** $(x^2-1)(x+3)$. **3.** $(a+1)(a+2)(2a+1)$. **4.** $(2p-1)(p+2)(p+8)$.

C. **1.** $(x-y)^2(x^2+xy+y^2)$. **2.** $x^2(x^2-y^2)$. **3.** $(3y+2)(y+2)(y+3)$. **4.** $(3b-7)(b+2)(b-2)$.

Exercise 50.

A. **1.** $(x^2-3x+4)\times(2x^2-x+2)\times(3x^2+x-2)$.
2. $(3x^2+x-1)\times(2x^2+2x+5)\times(4x^2+3x-5)$.
3. $(a^2+2a-3)\times(2x^2-3a-8)\times(2a^2-a-5)$.
4. $(2x+5)(x^2+3x+4)\times(x^2+2x+3)(x^2+x+2)$.

B. **1.** $(3y^2-2y+3)(y^2+y-1)\times(2y^2-y+1)$.
2. $(4a^2-a+3)(a^2+2a-4)\times(2a^2+a+6)$.
3. $3y^2(2x+y)(3x-y)(x+5y)$. **4.** $(a^2+3)(2a^2-a-1)\times(2a^2+a-2)\times(3a^2-a+2)$.

C. **1.** $(2x^2+x-4)(x^2+2x-3)\times(x^2+3x-2)$. **2.** $6a(3a-2)(3a-5)(a+7)$. **3.** $(y^2-3y+5)(2y^2+y-6)\times(6y^2-2y+3)$. **4.** $9(x^2-x+1)(3x+2)\times(x-1)(x-2)$.

Chap. XIII.—FRACTIONS.

Exercise 51.

A. **1.** $\frac{2y}{3x}$; $\frac{6c^2}{7a}$. **2.** $\frac{a}{a^2-2b^2}$; and $\frac{m-3n}{m^2+3mn+9n^2}$.
3. $\frac{3+x}{2}$. **4.** $\frac{y(x-4)}{x+5}$.

B. **1.** $\frac{3b}{4ac}$; $\frac{5m}{7n}$. **2.** $\frac{3a}{a+2}$; and $\frac{ab}{a-2}$. **3.** $\frac{m+7}{m+13}$.
4. $\frac{3(x+y)}{x-y}$.

C. **1.** $\frac{2x}{3a}$; $\frac{2z}{3xy}$ **2.** $\frac{a-1}{3a^2+3a+10}$; and $\frac{x-2y}{x+2y}$. **3.** $\frac{3x^2+y^2}{4x-y}$.
4. $\frac{m+5}{m+4}$.

Exercise 52.

A. **1.** $\frac{6a-1}{3}$: and $\frac{4xy+2}{y}$. **2.** $\frac{8a+3}{5}$: and $\frac{8y+4}{3}$.
3. $\frac{a^2+2b^2}{a+b}$. **4.** $\frac{3xy+3y^2}{x^2+2y}$.

B. **1.** $\frac{9m-2}{3}$: and $\frac{5xy+3}{x}$. **2.** $\frac{5ax-x^2}{2a}$: and $\frac{16y^2-2ay}{3y}$.
3. $\frac{1}{1+a}$. **4.** $\frac{x^2y^2+y^4}{x+y}$.

C. **1.** $\frac{20x-3}{4}$: and $\frac{3a^2+1}{a}$. **2.** $\frac{10by-6y-3y^2}{5b}$: and
$\frac{33x-4}{6}$. **3.** $\frac{1-2x+x^2}{1+x}$. **4.** $\frac{a^2-b^2-c^2-2bc}{a+b-c}$.

Exercise 53.

A. **1.** $2+\frac{5b}{4a}$. **2.** $2a-5x+\frac{3x^2}{2a}$. **3.** $7x-\frac{xy}{x+y}$.
4. $a^3-a^2x+ax^2-x^3+\frac{2x^4}{a+x}$.

B. **1.** $3+\frac{n}{m}$. **2.** $2a-3x-\frac{6x^2}{2a-3x}$. **3.** $3ax-x^2+\frac{3x^3}{2a+x}$.
4. $y-3+\frac{4}{y+1}$.

C. **1.** $5x+2y+\frac{y^2}{3x}$. **2.** $6x-a+\frac{3a^2}{3a-x}$. **3.** $a-1-\frac{5}{a-2}$.
4. $x^2-x+1-\frac{1}{x+1}$.

Exercise 54.

A. **1.** $\frac{4a+4}{5}$. **2.** $\frac{32x-29}{24}$. **3.** $\frac{9x-4z}{xz}$.
4. $\frac{a^3+18a^2-18a+18}{12a^3}$.

B. **1.** $\frac{18y+37}{18}$. **2.** $\frac{51a-25}{42}$. **3.** $\frac{5ab+7ac+7bc}{abc}$.
4. $\frac{6m^3+m^2-m+3}{m^3}$.

C. **1.** $\frac{23x-65}{24}$. **2.** $\frac{31y-27}{24}$. **3.** $\frac{41x-8y}{6xy}$.
4. $\frac{11x^3+36x^2-12x+10}{6x^3}$.

Exercise 55.

A. 1. $\frac{2a^2+12a+17}{a^2+7a+12}$. 2. $\frac{3x^4+2a^2x^2+3a^4}{x^4-a^4}$.

3. $\frac{91}{12x-24}$. 4. $\frac{33x+111}{(x+2)(x+3)(x+7)}$.

B. 1. $\frac{9x+21}{x^2+2x-15}$. 2. $\frac{2(x^4+y^4)}{x^4-y^4}$. 3. $\frac{43}{8a-12}$.

4. $\frac{22x-10}{(x^2-1)(x-2)}$.

C. 1. $\frac{4x}{1-x^4}$. 2. $\frac{3(a^2+b^2)}{a^2-b^2}$. 3. $\frac{10x+19y}{6x+12y}$.

4. $\frac{75x+41}{(2x+1)(2x+3)(3x-1)}$.

Exercise 56.

A. 1. $\frac{2a+7}{(a+4)(a+5)}$. 2. $\frac{3x-7}{(x-3)(x-4)}$.

3. $\frac{16x^2y}{(x+2y)(x-2y)^2}$. 4. $\frac{4(8x-1)}{3(1-4x^2)}$.

B. 1. $\frac{4mn}{4m^2-n^2}$. 2. $\frac{a+2b}{4a^2-9b^2}$. 3. $\frac{x^3+y^3+z^3-3xyz}{xyz}$.

4. $\frac{b^2-2ab-a^2}{ab(b^2-a^2)}$.

C. 1. $\frac{4x+9}{(2x+4)(2x+5)}$. 2. -1. 3. $\frac{a^4-2a^3b-2ab^3-b^4}{a^6-b^6}$.

4. $\frac{5}{6(a^2-9)}$.

Exercise 57.

A. 1. $\frac{7z+x-3xz}{x^2z}$. 2. $\frac{4x-19}{(x-3)(x-4)(x-7)}$. 3. $\frac{1}{a-3}$. 4. 0.

B. 1. $\frac{x+2}{(x-1)(2x+1)}$. 2. $\frac{4x+52}{(x-4)(x+3)(x+5)}$. 3. 0.

4. $\frac{2}{(x^2+a^2)(x^2+y^2)}$.

C. 1. $\frac{2}{a-b}$. 2. $\frac{6x}{x-a}$. 3. $\frac{y^4}{x^8-y^8}$. 4. 0.

Exercise 58.

A. **1.** 2. **2.** $\frac{a+2}{a+3}$. **3.** $\frac{a}{x-a}$. **4.** 1.

B. **1.** $\frac{ab}{2a-1}$. **2.** $\frac{y-11}{y-6}$. **3.** c. **4.** x.

C. **1.** $\frac{y-6}{y-3}$. **2.** $\frac{a+8}{a+4}$. **3.** $\frac{z-x+y}{z-x-y}$. **4.** $\frac{p-5}{p-1}$.

Exercise 59.

A. **1.** a^2+3a+9. **2.** $\frac{y+1}{y+5}$. **3.** $\frac{1}{x}$. **4.** $\frac{1}{3(2a+3)}$.

B **1.** $\frac{y-1}{4y+7}$. **2.** $\frac{(a-2b)(a+4)}{(a+2b)(a+5)}$. **3.** $\frac{x^4+y^4}{(x^2+y^2)^2}$.
4. $\frac{2}{3(x-2)}$.

C. **1.** $\frac{1}{m+7}$. **2.** $\frac{(5x+2)(2x^2+3x-5)}{(7x-4)(2x^2+3x+5)}$. **3.** $\frac{3x^2}{4a(x-2a)}$.
4. $\frac{1}{x^3+y^3}$.

Exercise 60.

	1.	**2.**	**3.**	**4.**
A.	$\frac{a(a+3)}{a+4}$.	$\frac{2(x+y)}{x-y}$.	$\frac{1}{2}$.	$\frac{a^6}{(a-c)(a+c)^2}$.
B.	$\frac{y+1}{y^2(y+3)}$.	$\frac{b^4}{b^2+a^2}$.	$\frac{x^2-x+1}{2x-1}$.	$\frac{xy}{x+y}$.
C.	$\frac{1}{b}$.	$\frac{1+y}{1+y^2}$.	$\frac{a^4}{1-a^8}$.	1.

Chap. XIV.—GENERAL TESTS.

Exercise 61.

A. **1.** $x+y$. **2.** $5a+3b-5c+8d$. **3.** $\frac{2}{3}(3a+4b)$. **4.** $\frac{4}{9}$.

5. (i) $b^4-4b^3+10b^2-12b+9$. (ii) $a^6-15a^4+75a^2-125$.

6. $\frac{x^2+y^2}{x^2-y^2}$.

B. **1.** $\frac{3}{2}a^2 - \frac{1}{3}b^2 + \frac{3}{4}c^2$. **2.** 1. **3.** (i) $a - \frac{3}{4}b$. (ii) $\frac{1}{2}a^2 - \frac{1}{8}ab - \frac{3}{16}b^2$. **4.** $\frac{a-3b}{a+3b}$. **5.** $y^2 - 1$. **6.** $\frac{x^4 + 2a^4}{a^2(x-a)^2(x^2+ax+a^2)}$.

C. **1.** 0. **2.** $\frac{17}{3}a + \frac{3}{4}b + \frac{32}{15}c$. **3.** $\frac{3}{4}a^5 - 4a^4 + \frac{77}{8}a^3 - \frac{43}{4}a^2 - \frac{33}{4}a + 27$. **4.** $\frac{3x^2 - x + 2}{3x^2 - 6x + 9}$. **5.** $(p+1)(p-1)(7p+2)(7p-2)$. **6.** $\frac{a+b}{a-b}$.

Exercise 62.

A. **1.** $52a - 48$. **2.** 14. **3.** $xy^2 - cx + cy$. **4.** —— **5.** $x - 2$. **6.** 0.

B. **1.** 1. **2.** $y^2 + (a-2c)y + a^2 + 3c^2$. **3.** 21. **4.** (i) $\frac{1}{4}x^2 + \frac{1}{9}y^2 + \frac{1}{16}z^2 + \frac{1}{3}xy - \frac{1}{4}xz - \frac{1}{6}yx$. (ii) $\frac{1}{8}a^3 + \frac{1}{4}a^2b + \frac{1}{6}ab^2 + \frac{1}{27}b^3$. **5.** $4a^2b^2$. **6.** $2x\left(\frac{x^2+y^2}{x^2-y^2}\right)^2$.

C. **1.** $-698a + 358b + 357c$. **2.** 884. **3.** $a(a^3+b^3)(3a-b)$ **4.** $2(a^2+b^2)(x^2+y^2)$. **5.** 0. **6.** $19a^2 + 37ab + 19b^2$.

Exercise 63.

A. **1.** $64\frac{2}{5}$. **2.** $-3x^2y - 5y^4$. **3.** (i) $(10y-1)(y+8)$. (ii) $(a-19b)(a+17b)$. **4.** $(2p+3)(4p+5)(3p-5) \times (p+2)(p-2)$. **5.** —— **6.** $\frac{2a}{a+5b}$.

B. **1.** —— **2.** (i) $(x+y)(x+y+1)$. (ii) $3a(a+9)(a-7)$. **3.** $\frac{2a^2}{8a^3-b^3}$. **4.** $\frac{5m^2-4m-8}{3m^2+4m-24}$. **5.** $4x^4 - 12x^3 - 7x^2 + 24x + 16$. **6.** $\frac{1}{(a-3)(a-4)}$.

C. **1.** 1152. **2.** $y - 5b$. **3.** $ab(a-b)$. **4.** $\frac{3a^2+1}{4a(a^2+1)}$. **5.** $\frac{1}{x^2} - \frac{1}{z^2}$. **6.** (i) $9a^4 - 12a^3 - 26a^2 + 20a + 25$. (ii) $\frac{1}{8}a^3 + \frac{3}{4}a^2 + \frac{3}{2}a + 1$.

Exercise 64.

A. **1.** —— **2.** (i) $(x^2+2xy-y^2)\times(x+y)$. (ii) $(x^2+2xy-y^2)\times(x-y)$. **3.** (i) $x^6-6x^5+13x^4-14x^3+10x^2-4x+1$. (ii) $\frac{a^3}{x^3}+3\frac{a}{x}+3\frac{x}{a}+\frac{x^3}{a^3}$. **4.** **G.C.M.** $7a-2b$. **L.C.M.** $(7a-2b)^2\times(7a+2b)(3a-2b)\times(3a+2b)$. **5.** 1. **6.** $4(x^2+1)(x-1)^2(x^2-2x-1)(x+1)^2$.

B. **1.** —— **2.** (i) $(2x-1)(x+3)(3x+7)$. (ii) $(a+b)(a^2-2ab+b^2)$. **3.** (i) $\frac{1}{9}x^4+\frac{2}{3}x^3+\frac{4}{3}x^2+x+\frac{1}{4}$. (ii) $8a^3+6a^2b+\frac{3}{2}ab^2+\frac{1}{8}b^3$ **4.** $\frac{c^2-a^2-b^2}{a^2-(b-c)^2}$. **5.** x^2+x+2. **6.** $2(x+y+z)$.

C. **1.** $1\frac{21}{30}$. **2.** p^2-2p-1. **3.** $x\frac{(2x+5y)(2x+7y)(x+6y)}{y(x-y)(x^2+2xy+2y^2)}$. **4.** (i) $9\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)-24\left(\frac{a}{b}+\frac{b}{a}\right)$. (ii) $\frac{1}{4}a^2(\frac{1}{2}a-b)+\frac{1}{3}b^2\times(\frac{1}{2}a-\frac{1}{9}b)$. **5.** (i) $(2a-b)(a-b)\times(a+b)$. (ii) $(x+y-1)(x^2+y^2-xy+x+y-1)$. **6.** $\frac{y}{x+y}$.

Exercise 65.

A. **1.** 6. **2.** $17\frac{6}{7}$. **3.** x^2-1. **4.** $5a^2+\frac{83}{24}ab-2b^2$. **5.** $\frac{1}{3}y^2+y-2$. **6.** $(a-1)(a+1)(2a+1)$, and $(x-1)(2x-1)(3x+1)$.

B. **1.** 4. **2.** 17. **3.** 0. **4.** $8x^4-2x^2-14x-5$. **5.** $\frac{1}{2}x^3-\frac{5}{3}x^2+\frac{1}{12}x+3$. **6.** $(2y-3)(y^2+y+1)$, and $(b-1)(b+3)(b-2)$.

C. **1.** 21. **2.** 4. **3.** 0. **4.** $x^2+\frac{4}{3}xy-\frac{65}{36}y^2-\frac{9}{2}y-\frac{9}{4}$. **5.** $1+4x-10x^2$. **6.** $(3x-1)(3x-2)$, and $(a-1)(a+2)(a^2-2a+4)$.

Exercise 66.

A. **1.** $\frac{a^2}{9a^2-16}$. **2.** $\frac{x-1}{x+1}$. **3.** $\frac{x}{z}+\frac{z}{n}-\frac{y}{m}$. **4.** $\frac{7x+2y}{7x+5y}$. **5.** $\frac{1}{a^6-1}$. **6.** $\frac{a^2+2a+3}{2a^3+3a^2+a-1}$.

B. 1. $\frac{2(3y+2)}{y+2}$. 2. $1+\frac{5a^2}{12}-\frac{5a^4}{36}-\frac{1}{16}a^6$. 3. $\frac{3}{4}x^2-\frac{2}{3}x-2$ 4. $\frac{2(a-3)}{a-2}$. 5. $\frac{2}{x+3}$. 6. ——.

C. 1. $m^2-2mn+4n^2$. 2. $\frac{a^2}{2}$. 3. $\frac{3x-4y}{-6x+y}$. 4. $\frac{1}{(x^2+1)(x^3+1)}$. 5. $\frac{a^2-b^2}{2(a^2+b^2)}$. 6. -40.

Exercise 67.

A. 1. $\frac{a}{(a^2+b^2)(x^2+b^2)(x+a)}$. 2. $\frac{x^3+y^3}{x(y^3-x^3)}$. 3. $2\sqrt{a^2-b^2}$. 4. $(x-y)(y-z)(z-x)$. 5. —. 6. —.

B. 1. $(a-b)(a-c)(a^2-bc.)$ 2. $\frac{1}{b-a}$. 3. $\frac{ax+by}{ax-by}$. 4. —— 5. 1. 6. $\frac{a^2}{(a+1)(a^2+1)}$.

C. 1. $(5.\overline{a+b}-2.\overline{x+y})\times(3.\overline{a+b}+4.\overline{x+y})$. 2. $\frac{b^3}{a^3}$. 3. $\frac{2}{b+c}$. 4. $\frac{2(x+c)}{(x-a)(x-b)}$. 5. $\frac{2}{a}(1+a^4)$. 6. $2\frac{1}{2}$.

Exercise 68.

A. 1. $-\frac{17}{6}$. 2. $1+5a+15a^2+45a^3$. 3. $y^2(4a-3y)(6a+y)$. 4. $\frac{6(b-a)}{b}$. 5. $12a^3(a-1)(a+1)\times(a+2)(a-3)$. 6. $\frac{x^2-y^2}{xy}$.

B. 1. 102. 2. $1-y-3y^2-y^3+5y^4$. 3. 1. 4. $2x^2-7x-3$. 5. $\frac{1}{4}x^2+\frac{1}{8}xz-\frac{1}{16}y^2+\frac{1}{64}z^2$. 6. $5a^2(3-2x)(5+x)$.

C. 1. $4y^2-8y+7$. 2. $\frac{1}{2}$. 3. $24x^3y^3$. 4. $(11x-21y^2).(x-3y^2)$. 5. $\frac{a-b}{4c}$. 6. $\frac{1-x}{1+x}$.

Exercise 69.

A. 1. $(ax+b)(x+1)$. **2.** $3x^2-2x+5$. **3.** 2

4. -49. **5.** a. **6.** $\frac{1}{a}$.

B. 1. $y-1-\frac{1}{y}$. **2.** $\frac{a^3+b^3+c^3}{abc}$. **3.** $(ax+by)(bx-ay)$. **4.** 1. **5.** $(2x^2-4x+3)(x^2+3x-1)$ and $(2x^2-4x+3)(x+2)(x-1)$. **6.** $\frac{a^2+2a+4}{a(a^3+1)}$.

C. 1. $3z-c$. **2.** $(ax+b)(bx-a)$. **3.** a^2. **4.** 1. **5.** $\frac{a(3a+y)}{2y^2(a+2y)}$. **6.** $\frac{25}{4}$.

Exercise 70.

A. 1. The 2nd is greater; difference=65280. **2.** $(2ax-by)(2bx-ay)$. **3.** —— **4.** 1. **5.** ·0953103. **6.** $\frac{a}{c}+\frac{ad+bc}{c^2}x+\frac{(ad+bc)dx^2}{c^3}+\frac{(ad+bc)d^2x^3}{c^4}+\ldots$

B. 1. $\frac{1}{2}$. **2.** $x(x-1)(7x+103)$. **3.** $x-y$. **4.** 1. **5.** $a^2b^2(3a^2-2ab+3b^2)$. **6.** $\left(\frac{x}{y}-\frac{y}{x}\right)^3=\frac{x^3}{y^3}-\frac{y^3}{x^3}-3\left(\frac{x}{y}-\frac{y}{x}\right)$.

C. 1. $(x+1)(ax-b)$. **2.** $2\left(\frac{bx}{ay}+\frac{ay}{bx}\right)$. **3. G.C.M.** $7a-2b$: **L.C.M.** $(7a-2b)^2(7a+2b)(3a-2b)(3a+2b)$. **4.** 1. **5.** $acx^2+(ad+bc)x+bd$, $acx^2+(bc-ad)x-bd$, $acx^2+(ad-bc)x-bd$, $acx^2-(ad+bc)x+bd$. **6.** -216.

THE END.